

# Cuts and Connectivity in Graphs and Hypergraphs

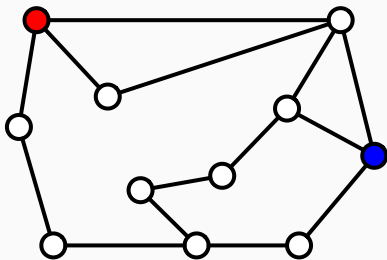
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Chao Xu

March 12, 2018

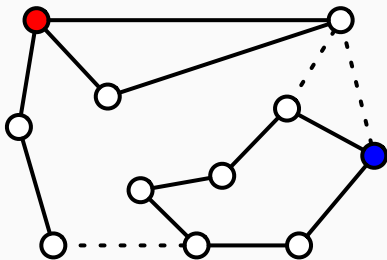
University of Illinois, Urbana-Champaign

## Min $st$ -cut in graphs



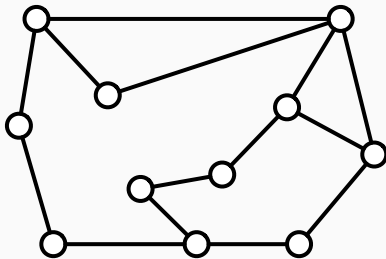
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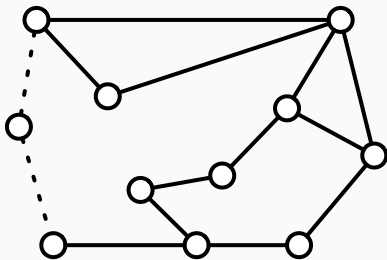
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## Min cut in graphs



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## Formal definition

### Problem: Min *st*-cut

**Input:**  $G = (V, E)$  and  $s, t \in V$

**Output:** A 2-partition  $(S, T)$  of the vertices  $V$ , such that  $s \in S$ ,  $t \in T$ , and the number of edges crossing  $S$  is minimized.

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Min *st*-cut problem is the fixed-terminal variant of min-cut problem, the global variant. The value of the min-cut is also called the (edge) connectivity, denoted  $\lambda$ .

## Applications of min-cut and min-*st*-cut

- Disconnect railroad networks,
- Maximum cardinality bipartite matching,
- Image segmentation,
- ...



## Algorithms for graph min-cut

- Finding a min-cut reduces to finding min- $st$ -cut for each pair of  $s$  and  $t$ .
- $\tilde{O}(mn)$  time: Maximum adjacency ordering. [Stoer-Wagner 95].
- $\tilde{O}(m)$  time (randomized). [Karger 98]
- $\tilde{O}(m)$  time (simple, unweighted). [Kawarabayashi-Thorup 15, Henzinger-Rao-Wang 17]

- Efficient algorithms for min-cut and its generalizations in graphs and hypergraphs.
- Understand the complexity difference between global and fixed-terminal variants.

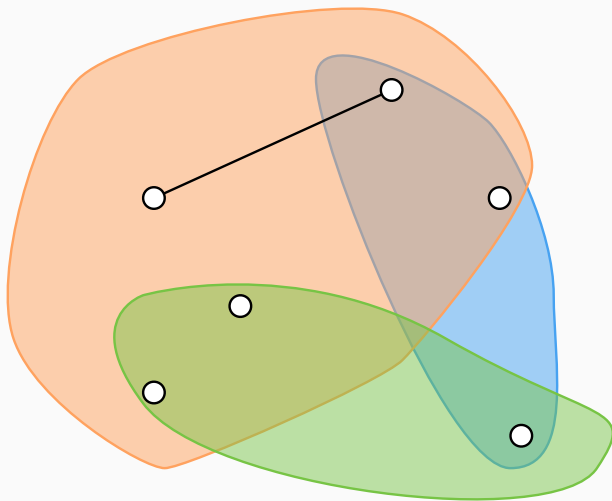
- Efficient algorithms for min-cut and its generalizations in graphs and hypergraphs.
- Understand the complexity difference between global and fixed-terminal variants.
  
- Algorithms for hypergraph min-cut.
- Approximation for bicut.
- Hypergraph  $k$ -cut.
- Minimum violation.

## Algorithms for hypergraph min-cut

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## A hypergraph

A **hypergraph**  $H = (V, E)$  consists of vertices  $V$  and edges  $E$ .

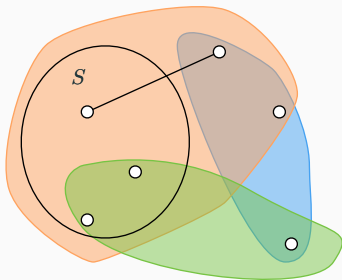


## Cut function

- $\delta_H(S)$  is the set of all edges cross  $S$
- The **cut function**  $c : 2^V \rightarrow \mathbb{N}$

$$c_H(S) = |\delta_H(S)|$$

- A set of vertices  $\emptyset \subsetneq S \subsetneq V$  is a min-cut if  $c_H(S)$  is minimized.



# Motivations for studying hypergraphs

- a natural generalization of graphs
- element connectivity [[Chekuri-Rukkanchanunt-X 16](#)]
- real life applications: VLSI, security, data mining, chemistry  
...

## The min-cut problems

- Find a min-cut.
- Find all min-cuts.
- Find a  $(1 + \epsilon)$ -approximate min-cut.



## Results on hypergraph cuts

	unweighted		weighted	
Problem	graph	hypergraph	graph	hypergraph
Min-cut	$O(m + \lambda n^2)$	$O(p + \lambda n^2)$	$\tilde{O}(mn)$	$\tilde{O}(pn)$
all min-cuts	$O(m + \lambda n^2)$	$O(p + \lambda n^2)$	$\tilde{O}(mn)$	$\tilde{O}(pn)$
$(1 + \epsilon)$ -min-cut	-	-	$O(m + n^2/\epsilon^2)$	$O(p + n^2 r^4/\epsilon^2)$

- $n$ : # vertices.
- $m$ : # edges.
- $p$ : sum of the cardinality of the edges.
- $\lambda$ : value of a min-cut.
- $r$ : the rank.

# Algorithms for hypergraph min-cut

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## Min-cut in unweighted hypergraphs

## Min-cut in unweighted graphs

1. Find a sparse subgraph with  $O(\lambda n)$  edges that preserves the min-cut in  $O(m)$  time.
2. Apply the  $O(mn)$  min-cut algorithm on the sparse subgraph.

Total running time:  $O(m + \lambda n^2)$

Subgraph  $H'$  a  $k$ -**certificate** of  $H$  if for all  $S \subseteq V$

$$c_{H'}(S) \geq \min(c_H(S), k).$$

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*Example:* a spanning tree is a 1-certificate of a connected graph.

**Theorem** ([Nagamochi-Ibaraki 92])

*A graph has a *k*-certificate with  $O(kn)$  edges, and can be found in  $O(m)$  time.*

## Application of $k$ -certificate

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- $(1 + \epsilon)$ -approximate min-cut.

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*Every hypergraph has a  $k$ -certificate with  $O(kn)$  edges (each edge can contain  $\Omega(n)$  vertices) and can be found in  $O(kp)$  time.*

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Running time  $O(\lambda p + \lambda n^3)$ .

**Theorem ([Chekuri-X 17])**

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Consequence:  $O(p + \lambda n^2)$  time algorithm for finding a min-cut.

# Algorithms for hypergraph min-cut

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All min-cuts



## What does it mean to find all min-cuts?

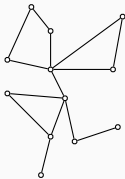
Find a small ( $O(n)$ -size) data structure that can quickly enumerate/count the number of min-cuts.

## A representation

A graph(hypergraph)  $G' = (V', E')$  is a **representation** of  $G = (V, E)$ , if there exists a function  $\phi: V \rightarrow V'$  such that

1.  $S'$  is a min-cut in  $G'$  iff  $\phi^{-1}(S')$  is a min-cut in  $G$ .
2.  $S$  is a min-cut in  $G$ , iff  $\phi(S)$  is a min-cut in  $G'$ .

## Graphs: Finding all min-cuts



A **cactus** is a graph in which any two cycles are edge disjoint.

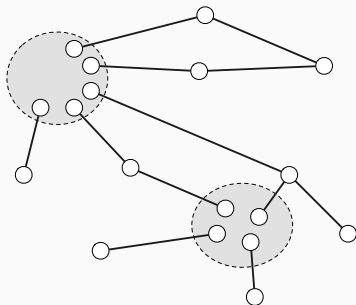
**Theorem** ([Dinitz et. al. 76, Karzanov & Timofeev 86])

*For each graph  $G$  there exist a representation  $G'$  where  $G'$  is a cactus.*

A cactus representation can be found

- in  $\tilde{O}(nm)$  time [Nagamochi et. al. 03]
- in (randomized)  $\tilde{O}(m)$  time [Karger & Panigrahi 09]

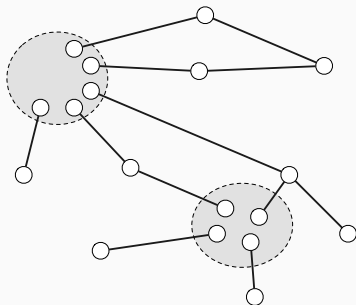
## Hypergraphs: Finding all min-cuts



**Theorem ([Cheng 99, Fleiner & Jordan 99])**

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### **Theorem ([Cheng 99, Fleiner & Jordan 99])**

*For each hypergraph  $H$  there exist a representation  $H'$  where  $H'$  is a hypercactus.*

Expensive to construct, in the order of  $O(n^4 p)$ .

## Hypergraphs: Finding all min-cuts

**Theorem ([Chekuri & X 17])**

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Approach: Using the decomposition framework [Cunningham 93].

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Matches the result in graphs. Conceptually simpler algorithm.



# Algorithms for hypergraph min-cut

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$(1 + \epsilon)$ -approximate min-cut

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1. Find a sparse subgraph that approximately preserves all min-cuts.
2. Find a min-cut in the sparse subgraph.

A graph  $G$  is a  $(1 \pm \epsilon)$ -**cut-sparsifier** of  $H$  if  
 $(1 - \epsilon)c_G(A) \leq c_H(A) \leq (1 + \epsilon)c_G(A)$  for all  $A \subseteq V$ .

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Too slow.

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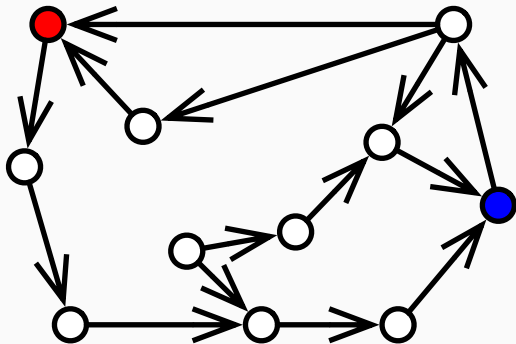
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Near-linear time when the hypergraph is dense.

**Summary: Fast hypergraph cut algorithms  
that match their state-of-the-art graph  
counterparts.**

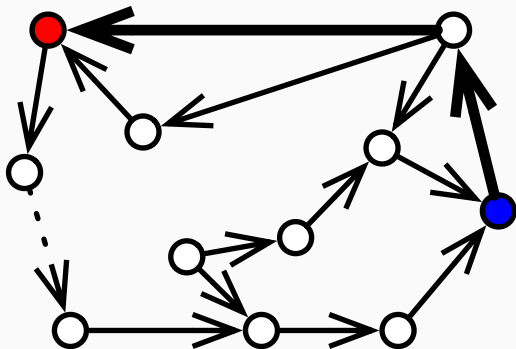
## Min-cut in directed graphs: Bicut

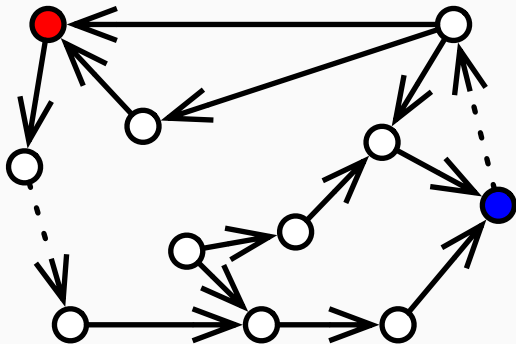
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## Bicut: Generalization of min-cut in directed graphs

- *st*-bicut: A set of edges such that its removal disconnect  $s$  and  $t$  in both direction.
- bicut: A *st*-bicut for some  $s$  and  $t$ .

A special case of multicut in directed graphs.

- Trivial 2-approximation. Union of min-*st*-cut and min-*ts*-cut [Dahlhaus et. al. 1994].
- $(2 - \epsilon)$ -inapproximable under UGC. [Chekuri & Madan 16, Lee 16]

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**Theorem ([Bérczi-Chandrasekaran-Király-Lee-X 17])**

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A hardness separation between fixed-terminal and global bicut!



## Vertex based interpretation of bicut

$A$  and  $B$  are **uncomparable** if  $A \setminus B \neq \emptyset$  and  $B \setminus A \neq \emptyset$ .

### Theorem

*The min-bicut problem is equivalent to two uncomparable sets  $A, B \subseteq V$  with minimum  $|\delta^{in}(A) \cup \delta^{in}(B)|$ .*

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Example: Find uncomparable sets  $A$  and  $B$  such that  $|\delta^{in}(A)| + |\delta^{in}(B)|$  is minimized. If it is not a  $(2 - \delta)$ -approximation, then most edges in the optimal bi-cut are going into  $A \cap B$ .

**Summary: A hardness gap between global and fixed-terminal bicut.**

## ***k*-cut in hypergraphs**

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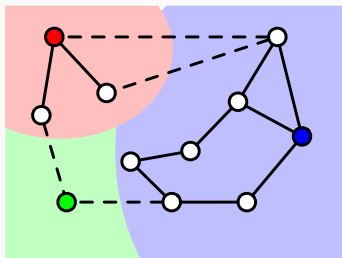
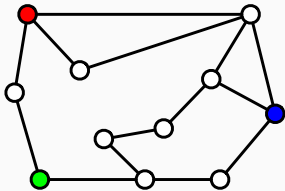
## $k$ -way-cut in graphs

### Problem: Min $k$ -way cut

**Input:**  $G$  and  $v_1, \dots, v_k \in V(G)$

**Output:** A  $k$ -partition  $(V_1, \dots, V_k)$ , such that  $v_i \in V_i$  for all  $i$ , and the number of edges crossing the partition classes is minimized.

A min- $k$ -cut is the minimum over all  $k$ -way-cut.



## Global vs. Fixed-terminal

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  - Divide and conquer:  $O(n^{(4+o(1))k})$  [Kamidoi-Yoshida-Nagamochi 07].
  - Divide and conquer:  $O(n^{(4-o(1))k})$  [Xiao 08].

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  - Tree packing:  $\tilde{O}(n^{2k})$  [Thorup 08].

**What about hypergraphs?**

## Previous works on HYPERGRAPH $k$ -cut

- Min  $k$ -way-cut is hard for  $k \geq 3$ .
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Main question: Hypergraph  $k$ -cut for  $k \geq 4$  in arbitrary rank hypergraphs?

Fixed-terminal vs. global complexity gap?



**Theorem ([Chandrasekaran-X-Yu 18])**

*There exists a randomized polynomial time algorithm that finds a minimum  $k$ -cut in a hypergraph.*

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Approach:

Randomized contraction algorithm with dampened sampling.

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**Theorem ([Chandrasekaran-X-Yu 18])**

*There are  $O(n^{2(k-1)})$  distinct min- $k$ -cuts in a hypergraph.*

**Summary: There is a global vs. fixed-terminal complexity gap for hypergraph  $k$ -cut.**

## Minimum violation

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## Violation

A **map** from  $G = (V, E)$  to  $H = (U, F)$  is a function  $f : V \rightarrow U$ .

$H$  is the **pattern graph**.

An edge  $uv \in E$  is a **violating edge**, if  $f(u)f(v) \notin F$ .

The **violation** of  $f$  is the number of violating edges.

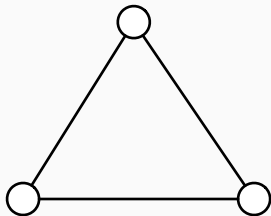
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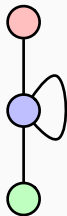
$H$  is the **pattern graph**.

An edge  $uv \in E$  is a **violating edge**, if  $f(u)f(v) \notin F$ .

The **violation** of  $f$  is the number of violating edges.



$G$



$H$

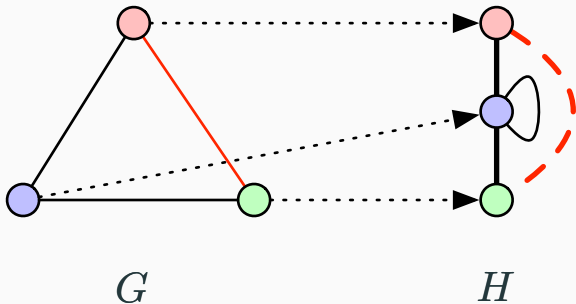
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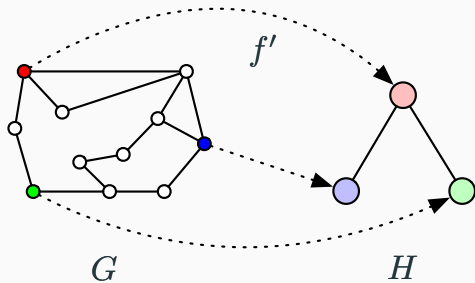


## Minimum violation retraction. $\text{RVio}(H)$

**Input:** graph  $G$  and a bijection  $f' : V' \rightarrow U$  for some  $V' \subseteq V(G)$

**Output:** A map  $f$  from  $G$  to  $H$  such that  $f|_{V'} = f'$  and the violation is minimized.

Vertices in  $V'$  are **fixed vertices**.

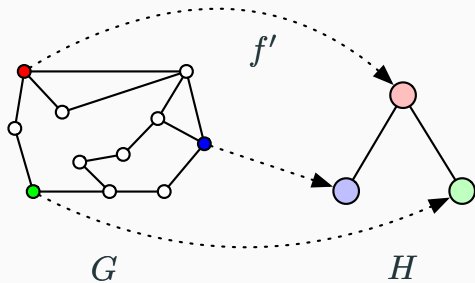


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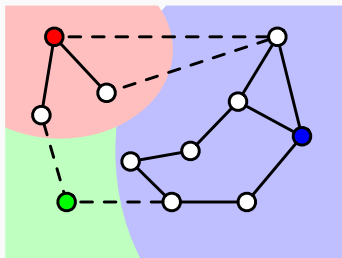
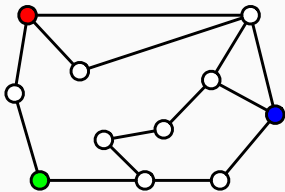
$H$  is **r-tractable** if  $\mathbf{RVio}(H)$  is tractable.

# $k$ -way cut

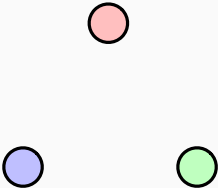
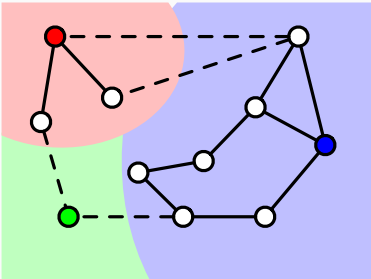
## Problem: Min $k$ -way cut

**Input:**  $G$  and  $v_1, \dots, v_k \in V(G)$

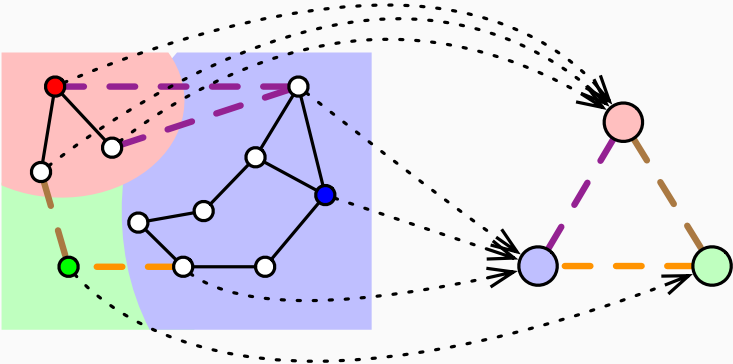
**Output:** A  $k$ -partition  $(V_1, \dots, V_k)$ , such that  $v_i \in V_i$  for all  $i$ , and the number of edges crossing the partition classes is minimized.



# 3-way cut



# 3-way cut



## Surjective Minimum Violation. $SVio(H)$

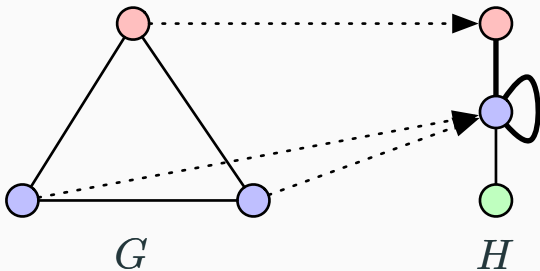
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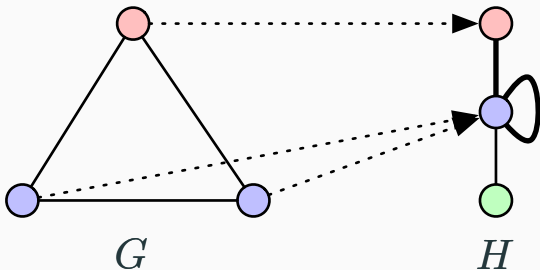


**NOT SURJECTIVE!**

## Surjective Minimum Violation. $SVio(H)$

**Input:**  $G = (V, E)$ .

**Output:** A **surjective** map from  $G$  to  $H$  with minimum violation.



**NOT SURJECTIVE!**  $H$  is **s-tractable** if  $SVio(H)$  is tractable.



## Why minimum violation?

Complete classification of  $r$ -tractable/ $s$ -tractable graphs implies complexity of various cut problems.

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Complete classification of  $r$ -tractable/ $s$ -tractable graphs implies complexity of various cut problems.

Classification of  $s$ -tractable graphs and  $r$ -tractable graphs was studied under the name “ $G_c$ -cut”. [[Elem-Hassin-Monnot 13](#)]

**Goal: classify the s-tractable and r-tractable graphs.**

# Classification of $r$ -tractable (directed) graphs

**Theorem** ([Kawarabayashi-X unpublished])

*There exists a polynomial time algorithm that decides if a (directed) graph is  $r$ -tractable.*

# Classification of $r$ -tractable graphs

$v$  dominates  $u$  if  $N(u) \subsetneq N(v)$ .

A graph  $G = (V, E)$  is a double-clique, if  $G = G[A] \cup G[B]$  for some clique  $A, B \subseteq V$ .

**Theorem** ([Kawarabayashi-X unpublished])

*A reflexive graph  $G$  is  $r$ -tractable if and only if  $G[U]$  is a double-clique, where  $U$  is the set of non-dominated vertices.*

## A theorem on $s$ -tractable graphs

**Theorem** ([Kawarabayashi-X unpublished])

*A reflexive graph  $H$  is  $s$ -tractable if and only if each of its component is  $s$ -tractable.*

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Consequences:

- $k$ -cut is solvable in polynomial time.
- Size-constrained  $k$ -cut: each partition class has at least  $c$  (a constant) vertices is solvable in polynomial time.



**Thank you!**