Cuts and Connectivity in Graphs and Hypergraphs

Chao Xu
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University of Illinois, Urbana-Champaign
Min \textit{st}-cut in graphs

Remove min number of edges to disconnect \textit{s} from \textit{t}.
Min $st$-cut in graphs

Remove min number of edges to disconnect $s$ from $t$. 
Min cut in graphs

Remove min number of edges to disconnect some pair of vertices.
Min cut in graphs

Remove min number of edges to disconnect some pair of vertices.
Formal definition

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Min \(st\)-cut problem is the fixed-terminal variant of min-cut problem, the global variant.
Formal definition

**Problem: Min \textit{st}-cut**

**Input:** $G = (V, E)$ and $s, t \in V$

**Output:** A 2-partition $(S, T)$ of the vertices $V$, such that $s \in S$, $t \in T$, and the number of edges crossing $S$ is minimized.

**Problem: Min-cut**

**Input:** $G = (V, E)$

**Output:** A 2-partition $(S, T)$ of the vertices $V$, such that the number of edges crossing $S$ is minimized.

Min \textit{st}-cut problem is the fixed-terminal variant of min-cut problem, the global variant. The value of the min-cut is also called the (edge) connectivity, denoted $\lambda$. 

6
Applications of min-cut and min-$st$-cut

- Disconnect railroad networks,
- Maximum cardinality bipartite matching,
- Image segmentation,
- ...
- Finding a min-cut reduces to finding min-st-cut for each pair of s and t.
- $\tilde{O}(mn)$ time: Maximum adjacency ordering. [Stoer-Wagner 95].
- $\tilde{O}(m)$ time (randomized). [Karger 98]
- $\tilde{O}(m)$ time (simple, unweighted). [Kawarabayashi-Thorup 15, Henzinger-Rao-Wang 17]
The Thesis

- Efficient algorithms for min-cut and its generalizations in graphs and hypergraphs.
- Understand the complexity difference between global and fixed-terminal variants.
The Thesis

- Efficient algorithms for min-cut and its generalizations in graphs and hypergraphs.
- Understand the complexity difference between global and fixed-terminal variants.

- Algorithms for hypergraph min-cut.
- Approximation for bicut.
- Hypergraph $k$-cut.
- Minimum violation.
Algorithms for hypergraph min-cut
A hypergraph $H = (V, E)$ consists of vertices $V$ and edges $E$. 
Cut function

- $\delta_H(S)$ is the set of all edges cross $S$
- The **cut function** $c : 2^V \rightarrow \mathbb{N}$

$$c_H(S) = |\delta_H(S)|$$

- A set of vertices $\emptyset \subsetneq S \subsetneq V$ is a min-cut if $c_H(S)$ is minimized.
Motivations for studying hypergraphs

- a natural generalization of graphs
- element connectivity [Chekuri-Rukkanchanunt-X 16]
- real life applications: VLIS, security, data mining, chemistry ...
The min-cut problems

- Find a min-cut.
- Find all min-cuts.
- Find a \((1 + \epsilon)\)-approximate min-cut.
## Results on hypergraph cuts

<table>
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<tr>
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<th>unweighted</th>
<th>weighted</th>
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<tr>
<td></td>
<td>graph</td>
<td>hypergraph</td>
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<tr>
<td>Min-cut</td>
<td>$O(m + \lambda n^2)$</td>
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- $n$: # vertices.
- $m$: # edges.
- $p$: sum of the cardinality of the edges.
- $\lambda$: value of a min-cut.
- $r$: the rank.
Algorithms for hypergraph min-cut

Min-cut in unweighted hypergraphs
Min-cut in unweighted graphs

1. Find a sparse subgraph with $O(\lambda n)$ edges that preserves the min-cut in $O(m)$ time.

2. Apply the $O(mn)$ min-cut algorithm on the sparse subgraph.

Total running time: $O(m + \lambda n^2)$
Subgraph $H'$ a $k$-certificate of $H$ if for all $S \subseteq V$

$$c_{H'}(S) \geq \min(c_H(S), k).$$
Subgraph $H'$ a **k-certificate** of $H$ if for all $S \subseteq V$

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*Example*: a spanning tree is a 1-certificate of a connected graph.
Subgraph $H'$ a \textbf{$k$-certificate} of $H$ if for all $S \subseteq V$

$$c_{H'}(S) \geq \min(c_H(S), k).$$

\textit{Example}: a spanning tree is a 1-certificate of a connected graph.

\textbf{Theorem ([Nagamochi-Ibaraki 92])}

A graph has a $k$-certificate with $O(kn)$ edges, and can be found in $O(m)$ time.
Application of $k$-certificate

- finding the connectivity.
Application of $k$-certificate

- finding the connectivity.
- spanning tree packing. [Gabow 98]
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- sketching in dynamic graph streams. [Guha-McGregor-Tench 15]
Application of $k$-certificate

- finding the connectivity.
- spanning tree packing. [Gabow 98]
- sketching in dynamic graph streams. [Guha-McGregor-Tench 15]
- $(1 + \epsilon)$-approximate min-cut.
$k$-certificate for hypergraphs?

1. Find a $\lambda$-certificate.
2. Apply the $O(pn)$ min-cut algorithm on the $\lambda$-certificate.
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**Theorem ([Guha-McGregor-Tench 15])**

*Every hypergraph has a $k$-certificate with $O(kn)$ edges (each edge can contain $\Omega(n)$ vertices) and can be found in $O(kp)$ time.*
1. Find a $\lambda$-certificate.
2. Apply the $O(pn)$ min-cut algorithm on the $\lambda$-certificate.

**Theorem ([Guha-McGregor-Tench 15])**

*Every hypergraph has a $k$-certificate with $O(kn)$ edges (each edge can contain $\Omega(n)$ vertices) and can be found in $O(kp)$ time.*

Running time $O(\lambda p + \lambda n^3)$. 
Theorem ([Chekuri-X 17])

Every hypergraph has a $k$-certificate with size $O(kn)$, and can be found in $O(p)$ time.
**Theorem ([Chekuri-X 17])**

Every hypergraph has a $k$-certificate with size $O(kn)$, and can be found in $O(p)$ time.

Consequence: $O(p + \lambda n^2)$ time algorithm for finding a min-cut.
Algorithms for hypergraph min-cut

All min-cuts
What does it mean to find all min-cuts?

Find a small \( O(n) \)-size data structure that can quickly enumerate/count the number of min-cuts.
A graph (hypergraph) $G' = (V', E')$ is a **representation** of $G = (V, E)$, if there exists a function $\phi: V \rightarrow V'$ such that

1. $S'$ is a min-cut in $G'$ iff $\phi^{-1}(S')$ is a min-cut in $G$.
2. $S$ is a min-cut in $G$, iff $\phi(S)$ is a min-cut in $G'$. 

A representation
A **cactus** is a graph in which any two cycles are edge disjoint.

**Theorem ([Dinitz et. al. 76, Karzanov & Timofeev 86])**

*For each graph $G$ there exist a representation $G'$ where $G'$ is a cactus.*

A cactus representation can be found

- in $\tilde{O}(nm)$ time [Nagamochi et. al. 03]
- in (randomized) $\tilde{O}(m)$ time [Karger & Panigrahi 09]
Theorem ([Cheng 99, Fleiner & Jordan 99])

For each hypergraph \( H \) there exist a representation \( H' \) where \( H' \) is a hypercactus.
Theorem ([Cheng 99, Fleiner & Jordan 99])

For each hypergraph $H$ there exist a representation $H'$ where $H'$ is a hypercactus.

Expensive to construct, in the order of $O(n^4 p)$. 
Theorem ([Chekuri & X 17])

A hypercactus representation can be found in $\tilde{O}(np)$ time.
Hypergraphs: Finding all min-cuts

**Theorem ([Chekuri & X 17])**

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**Approach:** Using the decomposition framework [Cunningham 93].
Theorem ([Chekuri & X 17])

*Hypercactus representation can be found in $\tilde{O}(np)$ time.*

Approach: Using the decomposition framework [Cunningham 93].
Matches the result in graphs. Conceptually simpler algorithm.
Algorithms for hypergraph min-cut

$(1 + \varepsilon)$-approximate min-cut
Finding an \((1 + \epsilon)\)-approximate min-cut

\[ S \text{ is a } (1 + \epsilon)\text{-approximate min-cut if } c(S) \leq (1 + \epsilon)\lambda. \]
Finding an \((1 + \epsilon)\)-approximate min-cut

\(S\) is a \((1 + \epsilon)\)-approximate min-cut if \(c(S) \leq (1 + \epsilon)\lambda\).

1. Find a sparse subgraph that approximately preserves all min-cuts.
2. Find a min-cut in the sparse subgraph.
A graph $G$ is a $(1 \pm \epsilon)$-cut-sparsifier of $H$ if

$$(1 - \epsilon)c_G(A) \leq c_H(A) \leq (1 + \epsilon)c_G(A)$$

for all $A \subseteq V$. 
Cut-sparsifiers

A graph $G$ is a $\left(1 \pm \epsilon\right)$-cut-sparsifier of $H$ if $(1 - \epsilon)c_G(A) \leq c_H(A) \leq (1 + \epsilon)c_G(A)$ for all $A \subseteq V$.

**Theorem ([Benczúr-Karger 98])**

There exists a $\left(1 \pm \epsilon\right)$-cut-sparsifier of $\tilde{O}(\frac{n}{\epsilon^2})$ edges, and can be constructed in $\tilde{O}(m)$ time with high probability.
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**Theorem ([Benczúr-Karger 98])**

There exists a $(1 \pm \epsilon)$-cut-sparsifier of $\tilde{O}(n^{1/2}/\epsilon^2)$ edges, and can be constructed in $\tilde{O}(m)$ time with high probability.

Consequence: A $(1 + \epsilon)$-approximate min-cut can be found in $\tilde{O}(m + n^2/\epsilon^2)$ time. Near-linear time when graph is dense.
**Theorem ([Kogan-Krauthgamer 14])**

There exists a $(1 \pm \epsilon)$-cut-sparsifier of $\tilde{O}\left(\frac{nr}{\epsilon^2}\right)$ edges, and can be constructed in $\tilde{O}(n^2 p)$ time with high probability.
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There exists a $(1 \pm \epsilon)$-cut-sparsifier of $\tilde{O}(\frac{nr}{\epsilon^2})$ edges, and can be constructed in $\tilde{O}(n^2 p)$ time with high probability.

Too slow.
Fast cut-sparsifiers in hypergraphs

**Theorem ([Chekuri-X 17])**

A $(1 \pm \epsilon)$-cut-sparsifier of $H$ with $\tilde{O}(nr^2/\epsilon^2)$ edges can be found in $\tilde{O}(p)$ time with high probability.
**Theorem ([Chekuri-X 17])**

A \((1 \pm \epsilon)\)-cut-sparsifier of \(H\) with \(\tilde{O}(nr^2/\epsilon^2)\) edges can be found in \(\tilde{O}(p)\) time with high probability.

Consequence: \(\tilde{O}(p + n^2 r^4/\epsilon^2)\) algorithm for \((1 + \epsilon)\)-approximate min-cut in hypergraphs.
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Consequence: $\tilde{O}(p + n^2r^4/\epsilon^2)$ algorithm for $(1 + \epsilon)$-approximate min-cut in hypergraphs. $\tilde{O}(p + n^2/\epsilon^2)$ for constant rank hypergraphs.
Fast cut-sparsifiers in hypergraphs

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Near-linear time when the hypergraph is dense.
Summary: Fast hypergraph cut algorithms that match their state-of-the-art graph counterparts.
Min-cut in directed graphs: Bicut
$st$-cut
Bicut: Generalization of min-cut in directed graphs

- **st-bicut**: A set of edges such that its removal disconnects $s$ and $t$ in both directions.
- **bicut**: A $st$-bicut for some $s$ and $t$. 
A special case of multicut in directed graphs.

- Trivial 2-approximation. Union of min-st-cut and min-ts-cut [Dahlhaus et. al. 1994].
- \((2 - \epsilon)\)-inapproximable under UGC. [Chekuri & Madan 16, Lee 16]
It is not known if computing bicut is NP-hard.
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**Theorem ([Bérczi-Chandrasekaran-Király-Lee-X 17])**

A \((2 - \delta)\)-approximation exists for min-bicut, where \(\delta = \frac{1}{448}\).
It is not known if computing bicut is NP-hard.

**Theorem ([Bérczi-Chandrasekaran-Király-Lee-X 17])**

A \((2 - \delta)\)-approximation exists for min-bicut, where \(\delta = \frac{1}{448}\).

A hardness separation between fixed-terminal and global bicut!
A and B are uncomparable if $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$.

**Theorem**

The min-bicut problem is equivalent to two uncomparable sets $A, B \subseteq V$ with minimum $|\delta^\text{in}(A) \cup \delta^\text{in}(B)|$. 
A and $B$ are uncomparable if $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$.

**Theorem**

*The min-bicut problem is equivalent to two uncomparable sets $A, B \subseteq V$ with minimum $|\delta^{in}(A) \cup \delta^{in}(B)|$.***

Approach: Find multiple relaxations such that one of them is a $(2 - \delta)$-approximation.
Vertex based interpretation of bicut

A and B are uncomparable if \( A \setminus B \neq \emptyset \) and \( B \setminus A \neq \emptyset \).

**Theorem**

The min-bicut problem is equivalent to two uncomparable sets \( A, B \subseteq V \) with minimum \(|\delta^\text{in}(A) \cup \delta^\text{in}(B)|\).

Approach: Find multiple relaxations such that one of them is a \((2 - \delta)\)-approximation.

Example: Find uncomparable sets \( A \) and \( B \) such that \(|\delta^\text{in}(A)| + |\delta^\text{in}(B)|\) is minimized. If it is not a \((2 - \delta)\)-approximation, then most edges in the optimal bi-cut are going into \( A \cap B \).
Summary: A hardness gap between global and fixed-terminal bicut.
$k$-cut in hypergraphs
**k-way-cut in graphs**

**Problem: Min k-way cut**

**Input:** \( G \) and \( v_1, \ldots, v_k \in V(G) \)

**Output:** A \( k \)-partition \((V_1, \ldots, V_k)\), such that \( v_i \in V_i \) for all \( i \), and the number of edges crossing the partition classes is minimized.

A min-\( k \)-cut is the minimum over all \( k \)-way-cut.
Global vs. Fixed-terminal

- Min $k$-way-cut is hard for $k \geq 3$. [Dahlhaus et. al. 94]
Global vs. Fixed-terminal

- Min $k$-way-cut is hard for $k \geq 3$. [Dahlhaus et. al. 94]
- Min $k$-cut. Multiple polynomial time algorithms!
  - Fix a partition class: $n^{\Theta(k^2)}$ [Goldschmidt-Hochbaum 94].
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  - Randomized contraction: $\tilde{O}(n^{2(k-1)})$ [Karger-Stein 96].
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  - Randomized contraction: $\tilde{O}(n^{2(k-1)})$ [Karger-Stein 96].
  - Divide and conquer: $O(n^{(4+o(1))k})$ [Kamidoi-Yoshida-Nagamochi 07].
  - Divide and conquer: $O(n^{(4-o(1))k})$ [Xiao 08].
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  - Divide and conquer: $O(n^{(4-o(1))k})$ [Xiao 08].
  - Tree packing: $\tilde{O}(n^{2k})$ [Thorup 08].
What about hypergraphs?
Previous works on HYPERGRAPH $k$-cut

- Min $k$-way-cut is hard for $k \geq 3$.
- Min $k$-cut.
  - $k = 2$: Hypergraph cut.
Previous works on HYPERGRAPH $k$-cut

- Min $k$-way-cut is hard for $k \geq 3$.
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  - Constant rank: Hypertree packing [Fukunaga 10].
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- Min $k$-cut.
  - $k = 2$: Hypergraph cut.
  - $k = 3$: Deterministic contraction [Xiao 08].
  - Constant rank: Hypertree packing [Fukunaga 10].

Main question: Hypergraph $k$-cut for $k \geq 4$ in arbitrary rank hypergraphs?

Fixed-terminal vs. global complexity gap?
Our result

**Theorem ([Chandrasekaran-X-Yu 18])**

There exists a randomized polynomial time algorithm that finds a minimum $k$-cut in a hypergraph.
Our result

**Theorem ([Chandrasekaran-X-Yu 18])**
There exists a randomized polynomial time algorithm that finds a minimum $k$-cut in a hypergraph.

Approach:
Randomized contraction algorithm with dampened sampling.
**Theorem ([Chandrasekaran-X-Yu 18])**

*There exists a randomized polynomial time algorithm that finds a minimum k-cut in a hypergraph.*

**Approach:**
Randomized contraction algorithm with dampened sampling.

**Theorem ([Chandrasekaran-X-Yu 18])**

*There are $O(n^{2(k-1)})$ distinct min-k-cuts in a hypergraph.*
Summary: There is a global vs. fixed-terminal complexity gap for hypergraph $k$-cut.
Minimum violation
A map from $G = (V, E)$ to $H = (U, F)$ is a function $f : V \to U$. 

$H$ is the pattern graph.

An edge $uv \in E$ is a violating edge, if $f(u)f(v) \not\in F$.

The violation of $f$ is the number of violating edges.
A map from $G = (V, E)$ to $H = (U, F)$ is a function $f : V \rightarrow U$.

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An edge $uv \in E$ is a violating edge, if $f(u)f(v) \notin F$.

The violation of $f$ is the number of violating edges.
Minimum violation retraction. $RVio(H)$

**Input:** graph $G$ and a bijection $f': V' \to U$ for some $V' \subseteq V(G)$

**Output:** A map $f$ from $G$ to $H$ such that $f|_{V'} = f'$ and the violation is minimized.

Vertices in $V'$ are **fixed vertices**.
Minimum violation retraction. $\text{RVio}(H)$

**Input:** graph $G$ and a bijection $f' : V' \rightarrow U$ for some $V' \subseteq V(G)$

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Vertices in $V'$ are **fixed vertices**.

$H$ is r-tractable if $\text{RVio}(H)$ is tractable.
Problem: Min $k$-way cut

Input: $G$ and $v_1, \ldots, v_k \in V(G)$

Output: A $k$-partition $(V_1, \ldots, V_k)$, such that $v_i \in V_i$ for all $i$, and the number of edges crossing the partition classes is minimized.
3-way cut
3-way cut
Surjective Minimum Violation. \text{SVio}(H)

\textbf{Input: } \ G = (V, E).

\textbf{Output: } A \textit{surjective} map from \ G \ to \ H \ with \ minimum \ violation.
Surjective Minimum Violation. SVio(H)

Input: \( G = (V, E) \).

Output: A surjective map from \( G \) to \( H \) with minimum violation.

NOT SURJECTIVE!
**Surjective Minimum Violation. \( \text{SVio}(H) \)**

**Input:** \( G = (V, E) \).

**Output:** A surjective map from \( G \) to \( H \) with minimum violation.

NOT SURJECTIVE! \( H \) is s-tractable if \( \text{SVio}(H) \) is tractable.
Complete classification of r-tractable/s-tractable graphs implies complexity of various cut problems.
Why minimum violation?

Complete classification of r-tractable/s-tractable graphs implies complexity of various cut problems.

Classification of s-tractable graphs and r-tractable graphs was studied under the name “$G_c$-cut”. [Elem-Hassin-Monnot 13]
Goal: classify the s-tractable and r-tractable graphs.
Classification of $r$-tractable (directed) graphs

**Theorem ([Kawarabayashi-X unpublished])**

There exists a polynomial time algorithm that decides if a (directed) graph is $r$-tractable.
\[ \nu \text{ dominates } u \text{ if } N(u) \subsetneq N(\nu). \]

A graph \( G = (V, E) \) is a double-clique, if \( G = G[A] \cup G[B] \) for some clique \( A, B \subseteq V \).

**Theorem ([Kawarabayashi-X unpublished])**

A reflexive graph \( G \) is \( r \)-tractable if and only if \( G[U] \) is a double-clique, where \( U \) is the set of non-dominated vertices.
A theorem on $s$-tractable graphs

**Theorem ([Kawarabayashi-X unpublished])**

A reflexive graph $H$ is $s$-tractable if and only if each of its component is $s$-tractable.
A theorem on $s$-tractable graphs

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A reflexive graph $H$ is $s$-tractable if and only if each of its component is $s$-tractable.

Consequences:

- $k$-cut is solvable in polynomial time.
A theorem on $s$-tractable graphs

**Theorem ([Kawarabayashi-X unpublished])**

A reflexive graph $H$ is $s$-tractable if and only if each of its component is $s$-tractable.

Consequences:

- $k$-cut is solvable in polynomial time.
- Size-constrained $k$-cut: each partition class has at least $c$ (a constant) vertices is solvable in polynomial time.
Thank you!