LP relaxation and Tree Packing for Minimum k-cuts

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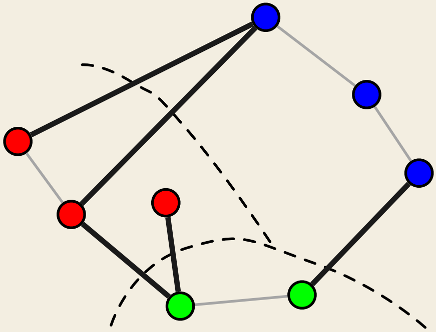
k-cut

Graph $G = (V, E)$. $c : E \to \mathbb{R}_+$ a capacity function. A *k-cut* is the set of edges crossing some partition of the vertices $\mathscr{P}$ such that $|\mathscr{P}| \geq k$. A *cut* is a 2-cut.



A min-k-cut is a k-cut with minimum capacity.

Applications

- Connectivity
- Image segmentation
- Clustering
- ...

Computation of min-cut

- Finding a min-cut reduces to finding min-st-cut for each pair of s and t.
- $\tilde{O}(mn)$ time: Maximum adjacency ordering. [Nagamochi-Ibaraki 92, Stoer-Wagner 95].
- $\tilde{O}(n^2 m)$ time: Randomized contraction. [Karger 92]
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Fastest algorithms are through tree packing.

Computation of min-k-cut

- Fix a partition class: $n^{\Theta(k^2)}$ [Goldschmidt-Hochbaum 94].
- Randomized contraction: $\tilde{O}(n^{2(k-1)})$ [Karger-Stein 96].
- Divide and conquer: $O(n^{(4+o(1))k})$ [Kamidoi-Yoshida-Nagamochi 07].
- Divide and conquer: $O(n^{(4-o(1))k})$ [Xiao 08].
- Tree packing : $\tilde{O}(n^{2k})$ [Thorup 08].
- Tree packing (and a lot of other ideas) :$O(Wk^{O(k)}n^{(1+\omega/3)k})$ randomized, $O(Wk^{O(k)}n^{(2\omega/3)k})$ deterministic [Gupta-Lee-Li 18]

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Main Property

For a set of edges A, a tree T h-respects A if $|T \cap A| \leq h$. All tree packing based min-cut algorithms shows the following theorem for some parameter of t, h, k.

Theorem

There exists a collection of t trees, such that for each min-k-cut A, there is a tree that h-respects A.

- Karger showed if $k = 2$, then $t = \tilde{O}(m)$ and $h = 1$.
- Thorup showed $t = \tilde{O}(mk^3)$ and $h = 2k - 2$.

Our contribution

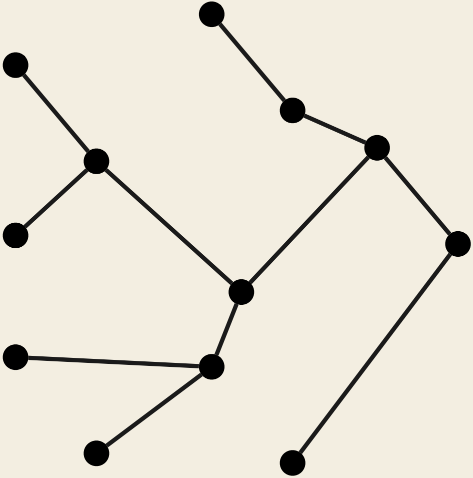
Analyzing the dual of k-cut LP [Naor and Rabani 01], to obtain a simple tree packing.

Theorem

There exists a collection of m trees, such that for each min-k-cut A, there is a tree that $(2k-3)$-respects A.

Implies a slightly faster deterministic algorithm for k-cut. $\tilde{O}(n^{2k-1})$.

A tree 3-respect a 3-cut



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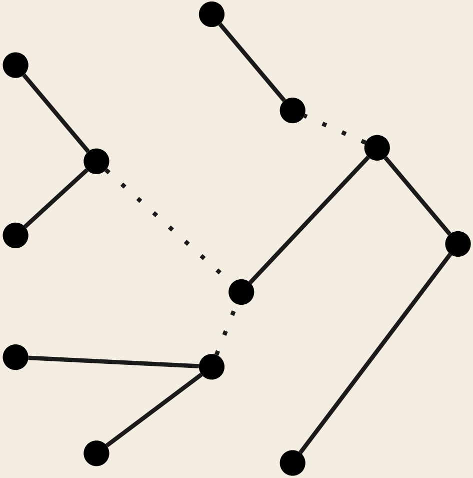

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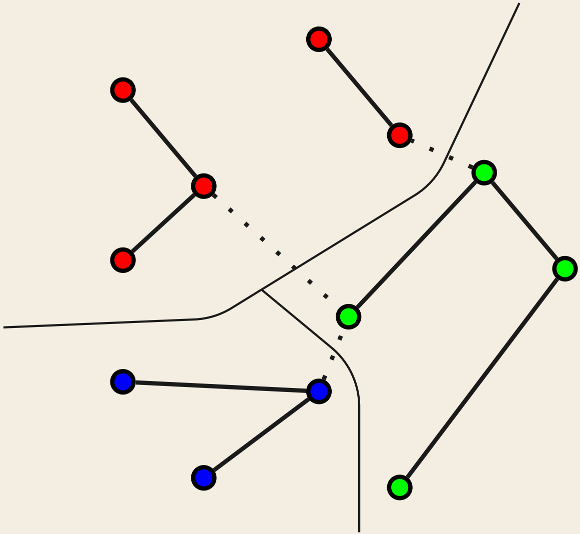

Algorithm (Same as Thorup)

Find all min-cuts given the collection of trees.

1. For each tree T in the collection, and set of $2k-3$ edges in T. Remove the edges, and group the obtained components into k parts. It is a candidate min-k-cut.
2. Return the candidates of the smallest value.

Running time $= (m \times \binom{n}{2k-3}$ set of edges $\times$ ways to partition $2k-2$ components into k parts).

$= O(mn^{2k-3}) = O(n^{2k-1})$

Tree packing and min-cut, a LP perspective

Cut LP'

$\mathcal{T}(G)$ is the set of spanning trees of G.

$$
\begin{aligned}
\min \sum_{e \in E} c_e x_e \\
\text{s.t.} \sum_{e \in T} x_e \geq 1 \text{ for all } T \in \mathcal{T}(G) \\
x_e \leq 1 \text{ for all } e \in E \\
x_e \geq 0 \text{ for all } e \in E
\end{aligned}
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c_e is positive, $x_e \leq 1$ is redundant.

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Theorem

The integrality gap of the cut LP is $2(1 - 1/n)$.

Tree packing LP

The *fractional spanning tree packing number*, $\tau(G)$, is the value of the following LP.

$$\begin{aligned} \max \sum_{T \in \mathscr{T}(G)} & y_T \\ s.t \quad \sum_{T \ni e} & y_T \leq c(e) \quad e \in E \\ & y_T \geq 0 \qquad T \in \mathscr{T}(G) \end{aligned}$$

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Tree packing LP is the dual of the Cut LP.
$\tau(G) \ge \frac{n}{2(n-1)} \cdot \lambda(G)$.

Tree packing and cuts

Theorem (Cut-Tree Packing Theorem)

Let y be a maximum tree packing. For each min-cut A, there exists a tree T in the packing that 1-respects A.

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Let y be a *maximum tree packing*. *For each min-cut A, there exists a tree T in the packing that 1-respects A.*

There exists a maximum tree packing consists of m trees.

Proof of the Cut-Tree Packing Theorem

Assume $\tau(G) = 1$. Otherwise we can scale all capacities by $c(e)/\tau(G)$.

y is a probability distribution over the spanning trees.

A is a fixed min-cut.

q is the fraction of trees that 1-respect A.

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q is the fraction of trees that 1-respect A. We want to show $q > 0$.

$$\sum_{T} y_T |T \cap A| \geq \sum_{T:|T\cap A|\geq 2} y_T |T \cap A| \geq 2(1-q).$$

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Let y be an ideal tree packing, such that each min-k-cut A, a constant faction of the trees $(2k-2)$-respects A.

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Let y be an ideal tree packing, such that each min-k-cut A, a constant faction of the trees $(2k-2)$-respects A.

$k = 2$, then $(2k-2) = 2$, a bit worse than the Cut-Tree Packing Theorem. The ideal tree packing consists of exponential number of trees. There is an approximate ideal tree packing with $\tilde{O}(mk^3)$ trees.

LP tree packing

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Let y be a maximum LP tree packing. For each min-k-cut A, there exists a tree in the packing that $(2k-3)$-respects A.

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A maximum LP tree packing is not a maximum tree packing of G.

The k-cut LP

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\begin{aligned}
\min \sum_{e \in E} c_e x_e & \\
\text{s.t.} \sum_{e \in T} x_e \geq k - 1 & \text{ for all } T \in \mathscr{T}(G) \\
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Theorem (Chekuri, Guha and Naor 06)

The integrality gap of the k-cut LP is $2(1-1/n)$.

Dual LP

$$\max\ (k-1)\sum_{T\in\mathscr{T}(G)} y_T - \sum_{e\in E} z_e$$

$$\text{s.t.} \sum_{T\ni e} y_T \leq c_e + z_e \text{ for all } e\in E$$

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The y in an optimal solution is called a maximum LP tree packing. y is NOT a tree packing under capacity c, but a tree packing for capacity $c + z$. z is called the extra capacity.

Theorem (k-Cut-Tree Packing Theorem)

Let y be a maximum LP tree packing. For each min-k-cut A, there exists a tree in the packing that $(2k-3)$-respects A.

Fix A min-k-cut A. Let q be the fraction of trees that $(2k-3)$-respects A. We will show that $q \geq \frac{1}{n}$.

Assume $\sum_T y_T = 1$.

$$(k-1)\sum_T y_T - z(E) \geq \frac{1}{2(1-\frac{1}{n})}\lambda_k(G)$$
$$k-1 \geq \frac{1}{2(1-\frac{1}{n})}\lambda_k(G) + z(E)$$

$$2\left(1-\frac{1}{n}\right)(k-1) \geq \lambda_k(G) + 2(1-1/n)z(E) \geq \lambda_k(G) + z(E).$$

$$\sum_T y_T|T \cap A| \geq \sum_{T:|T\cap A|\geq(2k-3)+1} y_T|T \cap A|$$
$$\geq 2(k-1)(1-q).$$

$$
\begin{aligned}
2(k-1)(1-q) &\leq \sum_T y_T |T \cap A| \\
&\leq c(A) + z(A) \\
&= \lambda_k(G) + z(A) \\
&\leq \lambda_k(G) + z(E) \\
&\leq 2(k-1)(1-1/n).
\end{aligned}
$$

$$
q \geq 1 - \frac{2(k-1)(1-\frac{1}{n})}{2(k-1)} = \frac{1}{n}
$$

Stronger statements

Theorem (Approximate k-Cut-Tree Packing Theorem)

Let y be a $(1-\varepsilon)$-approximate max LP tree packing. For each set of edges A such that $c(A) \leq \alpha\lambda_k(G)$,

- *If $\varepsilon = O(1/n)$, there exists a tree T that $(\lceil 2\alpha(k-1)\rceil - 1)$-respects A.*
- *If $\varepsilon = O(1/k)$, there is a constant faction of trees that $\lfloor 2\alpha(k-1)\rfloor$-respect A.*

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Corollary

There are $O(n^{\lfloor 2\alpha(k-1)\rfloor})$ α-approximate min-k-cuts.

Additional results

- A simple proof of the integrality gap of k-cut LP is $2(1 - \frac{1}{n})$.
- Explore the relation between Thorup's recursive tree packing, principal sequence of partitions, and Lagrangean relaxation approach to approximate k-cut [Barahona 00, Ravi and Sinha 08]