Hypergraph k-Cut in Randomized Polynomial Time

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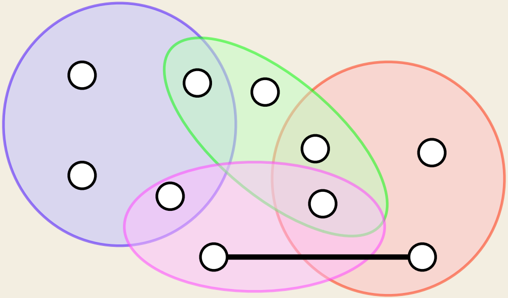
Hypergraph and *k*-cut



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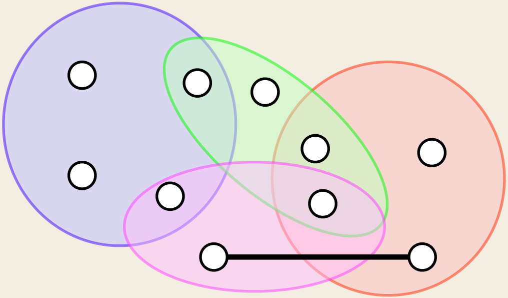

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Equivalently, set of edges whose removal disconnects the hypergraph into at least k components

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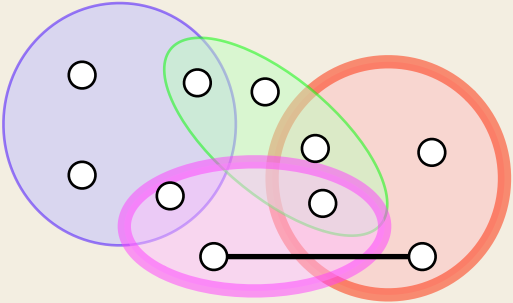

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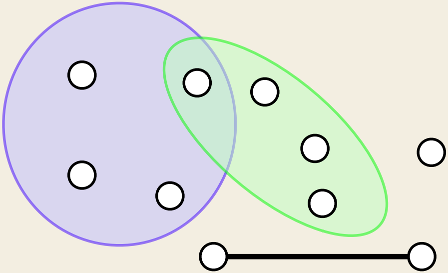

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The hypergraph k-cut problem

- Given: Hypergraph $G = (V, E)$
- Output: Minimum cardinality k-cut

Applications of k-cut

- Network reliability
- VLSI design
- Clustering
- ...

Previous works on GRAPH *k*-cut

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- Reduction to min st-cut using uncrossing arguments: $n^{\Theta(k^2)}$
  [Goldschmidt-Hochbaum 94]

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- Tree packing: $\tilde{O}(n^{2k})$ [Thorup 08]

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Hypergraph k-cut for $k \geq 4$ in arbitrary rank hypergraphs?

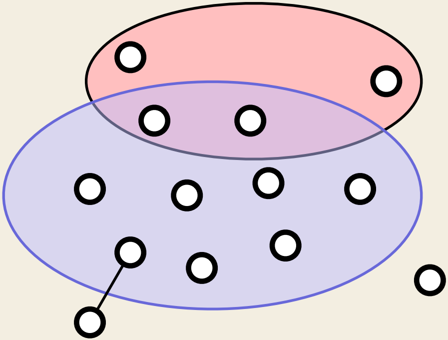
Our result

Theorem

There exists a randomized polynomial time algorithm to solve the hypergraph k-cut problem.

$k = 2$: Hypergraph cut (arbitrary rank)

Contractions in hypergraphs



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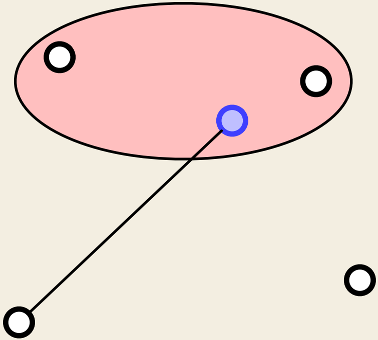
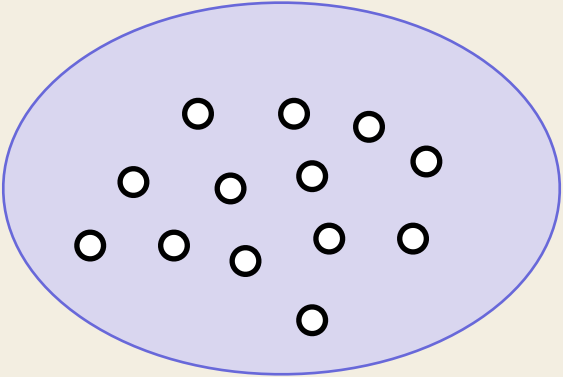
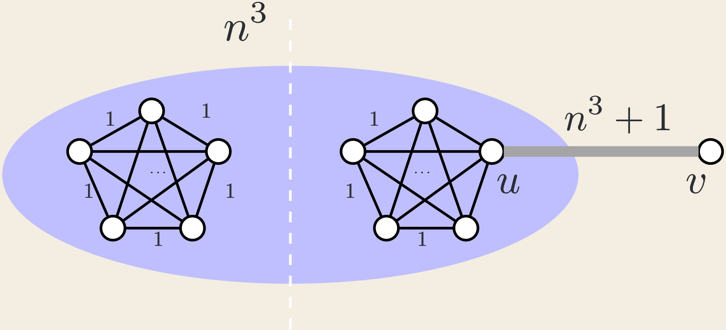


Edges in all cuts should not be contracted



Uniform probability contraction



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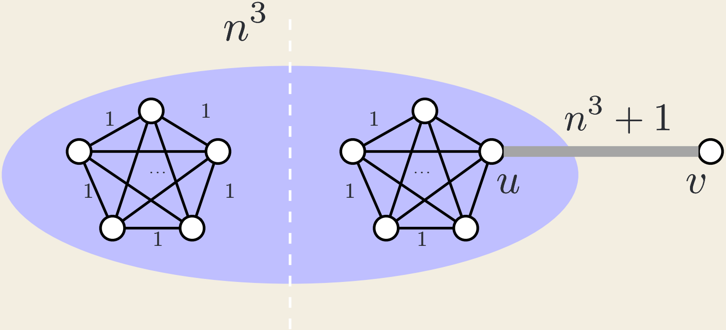

- Large probability of failure in a single step

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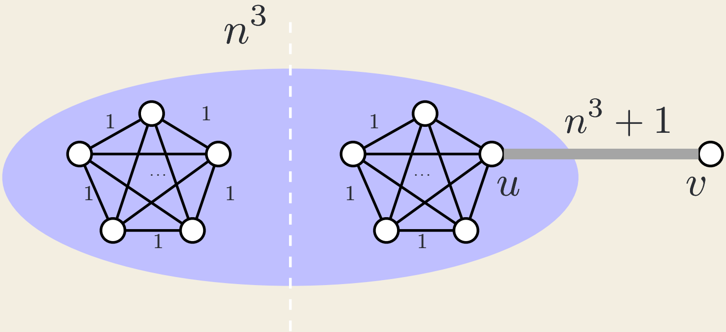

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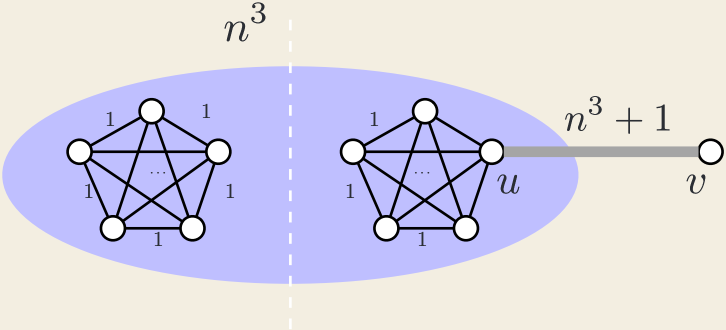

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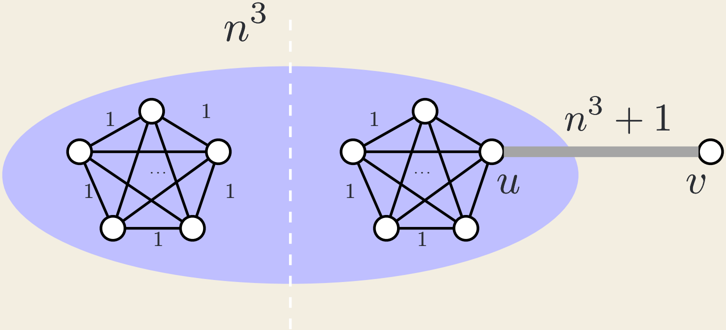

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- Destroys the min-cut with 1/2 probability
- 1/2 probability of success
- Unclear how to analyze

Our algorithm for hypergraph cut

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Input: Hypergraph G

While there are more than 4 vertices in G:

1. If $\sum_{e \in E} \delta_e = 0$, return E
2. Dampened sampling: Pick $e \in E$ with probability $p_e := \frac{\delta_e}{\sum_{f \in E} \delta_f}$
3. $G \leftarrow G/e$

Return a random min-cut in G by brute force

Analysis: Success probability

$$q_n := \min_{\substack{C^* \in OPT(G) \\ n \text{ node} \\ \text{hypergraph} G}} \Pr(\text{Algorithm returns } C^* \text{ on input } G)$$

Will show: $q_n \geq \frac{1}{\binom{n}{2}}$ by induction

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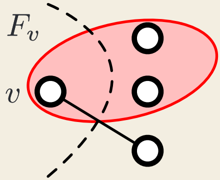

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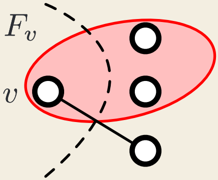

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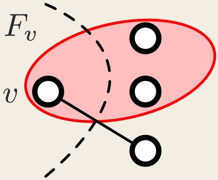

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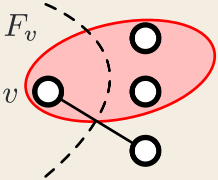

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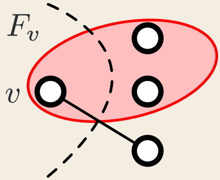

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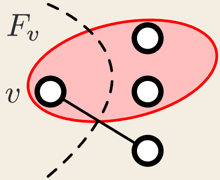

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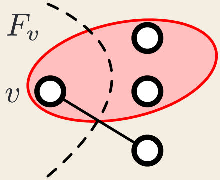

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Result

Theorem

The probability that the algorithm returns a particular min-cut is $\frac{1}{\binom{n}{2}}$. *Repeat it* $O(n^2 \log n)$ *times to obtain a min-cut with high probability.*

The algorithm for hypergraph k-cut

Similar algorithm as hypergraph cut, but different dampening.

$$\delta_e := \Pr_{S \sim \binom{V}{k-1}} (S \cap e = \emptyset) = \frac{\binom{n-|e|}{k-1}}{\binom{n}{k-1}}$$

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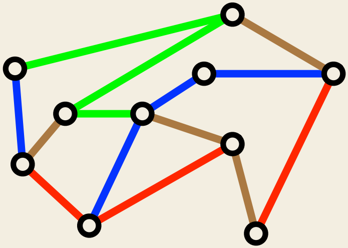
Corollary of our algorithm and analysis

The number of min-k-cuts is $O(n^{2(k-1)})$.

Additional Results: Hedgegraphs

Hedgegraphs



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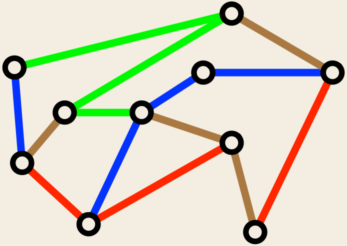

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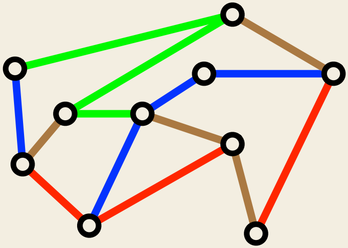
Hedgegraphs



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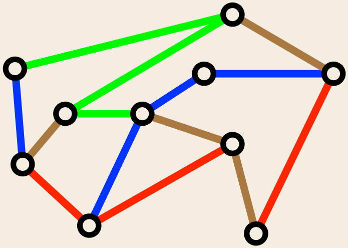

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- The underlying graph of a hedgegraph is the union of its hedges

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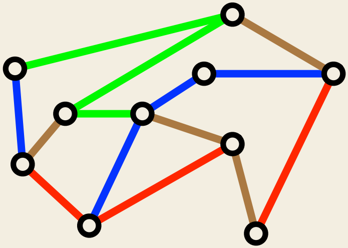

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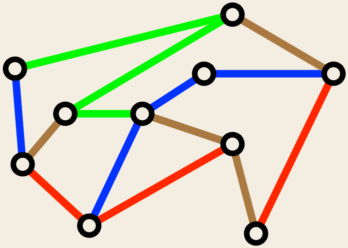
- The underlying graph of a hedgegraph is the union of its hedges
- Motivation: dependent edge failures

Hedgegraphs



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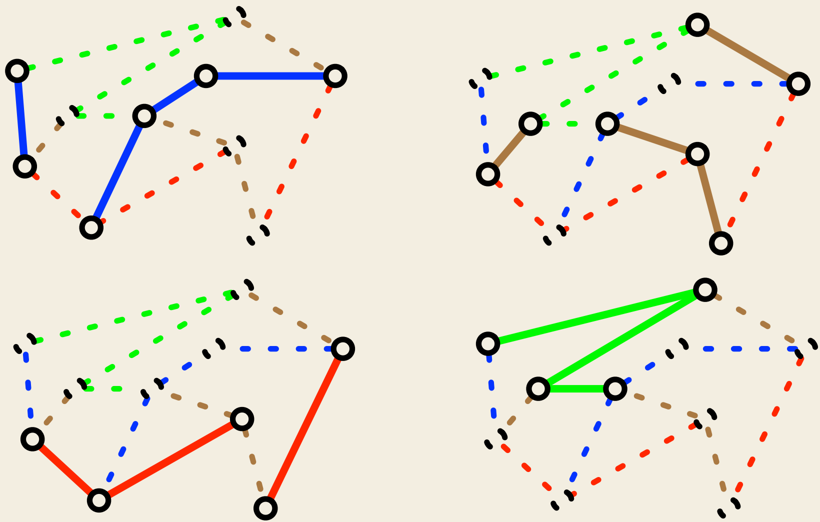
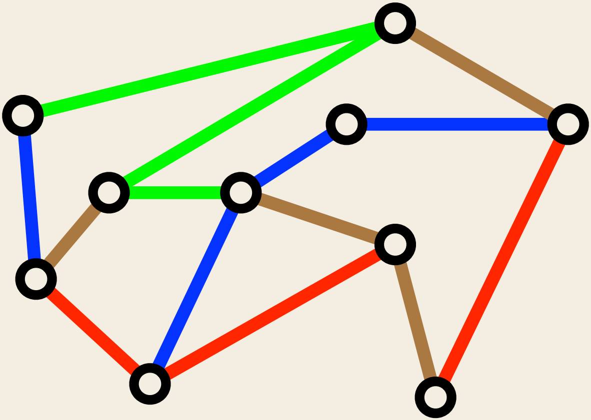
- Motivation: dependent edge failures
- Applications: layered networks, supply chain networks, ...

Hedges

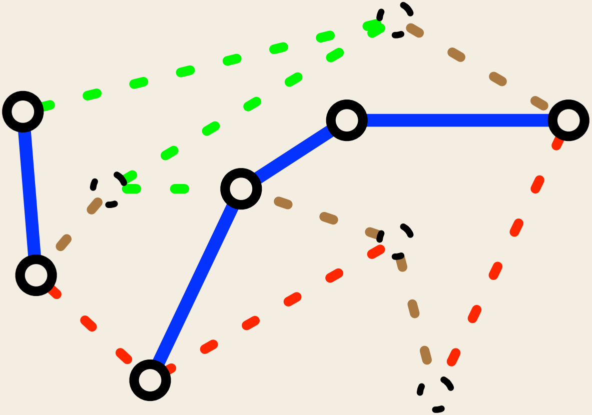


Span



The span of a hedge the number of components induced by a hedge

Span



Span of the blue hedge is 2

Additional results

- Poly-time algorithm for k-cut in constant span hedgegraphs (Hypergraphs are equivalent to hedgegraphs with span 1 [Ghaffari-Karger-Panigrahi 17])
- PTAS for k-cut in arbitrary span hedgegraphs

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Thank You!