Computing minimum cuts in hypergraphs*

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Abstract

We study algorithmic and structural aspects of connectivity in hypergraphs. Given a hypergraph $H = (V, E)$ with $n = |V|$, $m = |E|$ and $p = \sum_{e \in E} |e|$ the fastest known algorithm to compute a global minimum cut in H runs in $O(np)$ time for the uncapacitated case, and in $O(np + n^2 \log n)$ time for the capacitated case. We show the following new results.

- Given an uncapacitated hypergraph H and an integer k we describe an algorithm that runs in $O(p)$ time to find a subhypergraph H' with sum of degrees $O(kn)$ that preserves all edge-connectivities up to k (a k-sparsifier). This generalizes the corresponding result of Nagamochi and Ibaraki from graphs to hypergraphs. Using this sparsification we obtain an $O(p + \lambda n^2)$ time algorithm for computing a global minimum cut of H where λ is the minimum cut value.
- We generalize Matula's argument for graphs to hypergraphs and obtain a $(2+\varepsilon)$-approximation to the global minimum cut in a capacitated hypergraph in $O(\frac{1}{\varepsilon}(p \log n + n \log^2 n))$ time, and in in $O(p/\varepsilon)$ time for uncapacitated hypergraphs.
- We show that a hypercactus representation of *all* the global minimum cuts of a capacitated hypergraph can be computed in $O(np + n^2 \log n)$ time and $O(p)$ space.

Our results build upon properties of vertex orderings that were inspired by the maximum adjacency ordering for graphs due to Nagamochi and Ibaraki. Unlike graphs we observe that there are several different orderings for hypergraphs which yield different insights.

1 Introduction

We consider algorithmic and structural aspects of connectivity in hypergraphs. A hypergraph $H = (V, E)$ consists of a finite vertex set V and a set of hyperedges E where each edge e is a subset of vertices. Undirected loopless graphs are a special case of hypergraphs where all edges are sets of size two. For the most part we use n to denote the number of vertices $|V|$, m to denote the number of edges $|E|$, and p to denote $\sum_{e \in E} |e|$. Note that $p = \sum_{v \in V} \deg(v)$ where $\deg(v)$ is the degree of v (the number of hyperedges that contain v). We observe that p is the natural representation size of a connected hypergraph, and p is the number of edges in the standard representation of H as a bipartite graph $G_H = (V \cup E, F)$ with $F = \{(v, e) | v \in V, e \in E, v \in e\}$. A number of results on hypergraphs assume that the maximum edge size, often called the *rank*, is a fixed constant r. In this paper our focus is on general hypergraphs without assumptions on r.

Hypergraphs arise in a number of settings in both theory and practice. Some of the most basic algorithmic questions regarding hypergraphs have to do with connectivity and cuts. Given a hypergraph $H = (V, E)$, let $\delta_H(S)$ denote the set of edges that intersect both S and $V \setminus S$. It is well-known that the set function $|\delta_H(S)|$ defines a symmetric submodular set function over the ground set V. The connectivity (or the global minimum cut value) of a hypergraph H, denoted by $\lambda(H)$, is defined as $\min_{\emptyset \subsetneq S \subsetneq V} |\delta_H(S)|$; equivalently, it is the minimum number of edges that need to be removed such that H is disconnected. For distinct nodes $s, t \in V$ we denote by $\lambda_H(s, t)$ (or some times by $\lambda(s, t; H)$) the edge-connectivity between s and t in H which is defined as $\min_{S \subseteq V, |S \cap \{s,t\}| = 1} |\delta_H(S)|$. These definitions readily generalize to capacitated hypergraphs where each edge $e \in E$ has a non-negative capacity $c(e)$ associated with it. In this paper we study algorithmic and structural questions that arise in computing $\lambda(H)$. In the sequel we use the term mincut to refer to the global mincut.

Algorithms for mincuts and s-t mincuts in graphs have been extensively studied. Traditional algorithms for mincut were based on computing a sequence of $(n-1)$ s-t mincuts; s-t mincuts are most efficiently computed via network flow although one can also compute them via submodular function minimization. The first algorithm for

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finding a mincut in an *undirected* graph that avoided the use of flows was due to Nagamochi and Ibaraki [33]. They devised a surprising and influential algorithm based on maximum-adjacency orderings (MA-ordering) which is an ordering of the vertices based on a simple greedy rule. An MA-ordering can be computed in $O(m)$ time for uncapacitated graphs and in $O(m + n\log n)$ time for capacitated graphs. It has the following interesting property: if s and t are the last two vertices in the ordering then $\{t\}$ is an s-t mincut. This yields a simple $O(mn + n^2 \log n)$ time algorithm [38] for computing a mincut in a capacitated graph and is currently the asymptotically fastest *deterministic* algorithm. MA-orderings have other important structural properties which lead to several algorithmic and structural results — many of these are outlined in [34]. Karger devised another highly influential technique based on random contractions [20] which led to a randomized $O(n^2 \log^3 n)$-time Monte Carlo algorithm for computing a mincut in capacitated graphs [23]. Subsequently, using sampling techniques for cut-sparsification and tree packings, Karger devised a randomized $O(m \log^3 n)$ time Monte Carlo algorithm [21]. More recently Kawarabayashi and Thorup [24] devised a deterministic $O(m \log^{12} n)$ time algorithm for *simple* uncapacitated graphs.

What about hypergraphs? A simple and well-known reduction shows that $\lambda_H(s,t)$ can be computed via s-t network flow in the vertex capacitated bipartite graph G_H associated with H. Thus, using $(n-1)$ network flows one can compute $\lambda(H)$. However, Queyranne [36] showed that the Nagamochi-Ibaraki ordering approach generalizes to find the mincut of an arbitrary symmetric submodular function[1]. A specialization of the approach of Queyranne with a careful implementation leads to a deterministic $O(np + n^2 \log n)$-time algorithm for capacitated hypergraphs and an $O(np)$-time algorithm for uncapacitated hypergraphs. Two other algorithms achieving the same run-time were obtained by Klimmek and Wagner [25] and Mak and Wong [30]. Both these algorithms are based on the Nagamochi and Ibaraki ordering approach. Surprisingly, the orderings used by these three algorithms can be different for the same hypergraph even though they are identical for graphs[2]! We will later show that we can exploit their different properties in our algorithms.

Apart from the above mentioned results, very little else is known in the algorithms literature on mincut and related problems in hypergraphs despite several applications, connections, and theoretical interest. Recent work has addressed streaming and sketching algorithms when the rank is small [19,26]. Our initial motivation to address these algorithmic questions came from the study of algorithms for element-connectivity and related problems which are closely related to hypergraphs — we refer the reader to the recent survey [4]. In this paper the two main questions we address are the following.

- Are there faster (approximation) algorithms for mincut computation in hypergraphs?
- How many distinct mincuts can there be? Can a compact representation called the hypercactus that is known to exist [6, 10] be computed fast? For graphs it is well-known that there are at most $\binom{n}{2}$ mincuts and that there exists a compact $O(n)$-sized data structure called the *cactus* to represent all of them.

1.1 Overview of Results

In this paper we address the preceding questions and obtain several new results that we outline below.

Sparsification and fast algorithm for small mincuts: A k-sparsifier of a graph $G = (V,E)$ is a sparse subgraph $G' = (V,E')$ of G that preserves all local connectivities in G up to k; that is $\lambda_{G'}(s,t) \geq \min\{k, \lambda_G(s,t)\}$ for all $s,t \in V$. Nagamochi and Ibaraki [33] showed, via MA-ordering, that a k-sparsifier with $O(kn)$ edges can be found in linear time. In the hypergraph setting, a k-sparsifier is a subhypergraph preserving local connectivity up to k. A k-sparsifier with $O(kn)$ edges exists by greedy spanning hypergraph packing [19]. However, the sum of degrees in the sparsifier can be $O(kn^2)$. Indeed, any k-sparsifier through edge deletion alone cannot avoid the n^2 factor. We consider a more general operation where we allow *trimming* of hyperedges; that is, a vertex $v \in e$ can be removed from e without e itself being deleted. Trimming has been used for various connectivity results on hypergraphs. For example, in studying k-partition-connected hypergraphs, or in extending Edmonds' arborescence packing theorem to directed hypergraphs [14] (see [12, Section 7.4.1, Section 9.4] for a exposition of the results using the trimming terminology).

We show that for any hypergraph H on n nodes there is a k-sparsifier H' that preserves all the local connectivities up to k such that the size of H' in terms of the sum of degrees is $O(kn)$. In fact the sparsifier has the stronger property that all cuts are preserved up to k: formally, for any $A \subseteq V$, $|\delta_{H'}(A)| \geq \min\{k, |\delta_H(A)|\}$. Moreover such

[1] For a submodular function $f : 2^V \to \mathbb{R}$ the mincut is defined naturally as $\min_{\emptyset \subsetneq S \subsetneq V} f(S)$.
[2] This observation does not appear to have been explicitly noted in the literature.

a sparsifier can be constructed in $O(p)$ time. This leads to an $O(p+\lambda n^2)$ time for computing the mincut in an uncapacitated hypergraph, substantially improving the $O(np)$ time when λ is small and p is large compared to n. Sparsification is of independent interest and can be used to speed up algorithms for other cut problems.

$(2+\varepsilon)$ **approximation for global mincut:** Matula [31], building on the properties of MA-ordering, showed that a $(2+\varepsilon)$ approximation for the global mincut of an uncapacitated graph can be computed in deterministic $O(m/\varepsilon)$ time. The algorithm generalizes to capacitated graphs and runs in $O(\frac{1}{\varepsilon}(m\log n + n\log^2 n))$ time (as mentioned by Karger [20]). Although the approximation is less interesting in light of the randomized $\tilde{O}(m)$ algorithm of Karger [21], it is a useful building block that allows one to deterministically estimate the value of a mincut. For hypergraphs there was no such approximation known. In fact, the survey [4] observed that a near-linear time randomized $O(\log n)$-approximation follows from tree packing results, and raised the question of whether Matula's algorithm can be generalized to hypergraphs[3].

In this paper we answer the question in the affirmative and obtain a $(2+\varepsilon)$-approximation algorithm for the mincut of a capacitated hypergraph that runs in near-linear time — more formally in $O(\frac{1}{\varepsilon}(p\log n + n\log^2 n))$ time. For a uncapacitated hypergraph, the algorithm runs in $O(p/\varepsilon)$ time.

All mincuts and hypercactus: Our most technical contribution is for the problem of finding all the mincuts in a hypergraph. For any capacitated graph G on n vertices, it is well-known, originally from the work of Dinitz, Karzanov and Lomonosov [9], that there is a compact $O(n)$ sized data structure, namely a cactus graph, that represents all the mincuts of G. A cactus is a connected graph in which each edge is in at most one cycle (can be interpreted as a tree of cycles). As a byproduct one also obtains the fact that there are at most $\binom{n}{2}$ distinct mincuts in a graph; Karger's contraction algorithm gives a very different proof. After a sequence of improvements, there exist deterministic algorithms to compute a cactus representation of the mincuts of a graph in $O(mn + n^2\log n)$ time [32] or in $O(nm\log(n^2/m))$-time [11, 17]. For uncapacitated graphs, there is an $O(m + \lambda^2 n\log(n/\lambda))$-time algorithm [17]. There is also a Monte Carlo algorithm that runs in $\tilde{O}(m)$ time [22] building on the randomized near-linear time algorithm of Karger [21]. In effect, the time to compute the cactus representation is the same as the time to compute the global mincut! We note, however, that all the algorithms are fairly complicated, in particular the deterministic algorithms.

The situation for hypergraphs is not as straight forward. First, how many distinct mincuts can there be? Consider the example of a hypergraph $H=(V,E)$ with n nodes and a single hyperedge containing all the nodes. Then it is clear that every $S\subseteq V$ with $1\leq |S| < |V|$ defines a mincut and hence there are exponentially many. However, all of them correspond to the same edge-set. A natural question that arises is whether the number of distinct mincuts in terms of the edge-sets is small. Indeed, one can show that it is at most $\binom{n}{2}$. However, this fact does not seem to have been explicitly mentioned in the literature although it was known to some experts. It is a relatively simple consequence of fundamental decomposition theorems of Cunningham and Edmonds [8], Fujishige [15], and Cunningham [7] on submodular functions from the early 1980s. Cheng, building on Cunningham's work [7], explicitly showed that the mincuts of a hypergraph admit a compact hypercactus structure. Later Fleiner and Jordan [10] showed that such a structure exists for any symmetric submodular function defined over crossing families. However, these papers were not concerned with algorithmic considerations.

In this paper we show that the hypercactus representation of the mincuts of a hypergraph, a compact $O(n)$ sized data structure, can be computed in $O(np + n^2\log n)$ time and $O(p)$ space. This matches the time to compute a single mincut. The known algorithms for cactus construction on graphs are quite involved and directly construct the cactus. We take a different approach. We use the structural theory developed in [6, 7] to build the canonical decomposition of a hypergraph which then allows one to build a hypercactus easily. The algorithmic step needed for constructing the canonical decomposition is conceptually simpler and relies on an efficient algorithm for finding a non-trivial mincut (one in which both sides have at least two nodes) in a hypergraph H if there is one. Our main technical contribution is to show that there is an algorithm for finding a slight weakening of this problem that runs in $O(p + n\log n)$ time. Interestingly, we establish this via the ordering from the paper of [30]. Our algorithm yields a conceptually simple algorithm for graphs as well and serves to highlight the power of the decomposition theory for graphs and submodular functions [7, 8, 15].

1.2 Other Related Work

In a recent work Kogan and Krauthgamer [26] examined the properties of random contraction algorithm of Karger when applied to hypergraphs. They showed that if the rank of the hypergraph is r then the number of α-mincuts

[3]We use near-linear-time to refer to algorithms that run in time $O(p\log^c n)$ for some fixed constant c.

for $\alpha \geq 1$ is at most $O(2^{\alpha r} n^{2\alpha})$ which is a substantial improvement over a naive analysis that would give a bound of $O(n^{r\alpha})$. The exponential dependence on r is necessary. They also showed cut-sparsification results ala Benczur and Karger's result for graphs [3]. In particular, given a n-vertex capacitated hypergraph $H = (V, E)$ of rank r they show that there is a capacitated hypergraph $H' = (V, E')$ with $O(\frac{n}{\varepsilon^2}(r + \log n))$ edges such that every cut capacity in H is preserved to within a $(1 \pm \varepsilon)$ factor in H'. Aissi et al. [1] considered parametric mincuts in graphs and hypergraphs of fixed rank and obtained polynomial bounds on the number of distinct mincuts.

Hypergraph cuts have also been studied in the context of k-way cuts. Here the goal is to partition the vertex set V into k non-empty sets so as to minimize the number of hyperedges crossing the partition. For $k \leq 3$ a polynomial time algorithm is known [40] while the complexity is unknown for fixed $k \geq 4$. The problem is NP-Complete when k is part of the input even for graphs [18]. Fukunaga [16] obtained a polynomial-time algorithm for k-way cut when k and the rank r are fixed; this generalizes the result the polynomial-time algorithm for graphs [18, 39]. Karger's contraction algorithm also yields a randomized algorithm when k and the rank r are fixed. When k is part of the input, k-way cut in graphs admits a $2(1-1/k)$-approximation [37]. This immediately yields a $2r(1-1/k)$-approximation for hypergraphs. If r is not fixed and k is part of the input, it was recently shown [5] that the approximability of the k-way cut problem is related to that of the k-densest subgraph problem.

Hypergraph cuts arise in several other contexts with terminals such as the s-t cut problem or its generalizations such as multi-terminal cut problem and the multicut problem. In some of these problems one can reduce the hypergraph cut problem to a node-capacitated undirected graph cut problem and vice-versa.

2 Preliminaries

A hypergraph $H = (V, E)$ is capacitated if there is a non-negative edge capacity function $c : E \to \mathbb{R}_+$ associated with it. If all capacities are 1 we call the hypergraph uncapacitated; we allow multiple copies of an edge in the uncapacitated case. A cut $(S, V - S)$ is a bipartition of the vertices, where S and $V - S$ are both non-empty. We will abuse the notation and call a set S a cut to mean the cut $(S, V - S)$. Two sets A and B *cross* if $A \cap B$, $A \setminus B$ and $B \setminus A$ are all non-empty. For $S \subseteq V$, $\delta_H(S)$ is defined to be the set of all edges in $E(H)$ that have an endpoint in both S and $V - S$. We will drop H from the notation if the hypergraph is clear from the context. The capacity of the cut S, denoted by $c(S)$, is defined to be $\sum_{e \in \delta(S)} c(e)$. $\lambda(H)$ is the connectivity of H and is defined as $\min_{\emptyset \subsetneq S \subsetneq V} c(S)$. A cut S is a mincut of H if $c(S) = \lambda(H)$. A set of edges F is an edge-cut-set if $F = \delta(S)$ for some cut S. A set of edges is a min edge-cut-set if $F = \delta(S)$, where S is a mincut. For distinct vertices $s, t \in V(H)$, an s-t cut is a cut S such that $|S \cap \{s, t\}| = 1$. $\lambda(s, t; H)$ is the value of the s-t mincut(minimum s-t cut) in H. $\mathsf{sum\text{-}deg}(H)$ is defined as $\sum_{e \in E} |e|$ in the uncapacitated case and as $\sum_{e \in E} |e| c(e)$ in the capacitated case.

For pairwise disjoint vertex subsets $A_1, \ldots, A_k$, $E(A_1, \ldots, A_k; H) = \{e \mid e \cap A_i \neq \emptyset, 1 \leq i \leq k\}$ is the set of edges that have an end point in each of the sets $A_1, \ldots, A_k$. $d(A_1, \ldots, A_k; H) = \sum_{e \in E(A_1, \ldots, A_k; H)} c(e)$ denotes the total capacity of the edges in $E(A_1, \ldots, A_k; H)$. A related quantity for two disjoint sets A and B is $d'(A, B; H) = \sum_{e \in E(A,B;H), e \subseteq A \cup B} c(e)$ where only edges completely contained in $A \cup B$ are considered. As before, if H is clear from the context we drop it from the notation.

Removing a vertex $v \in e$ from e is called *trimming* e [12]. A hypergraph $H' = (V', E')$ is a *subhypergraph* of $H = (V, E)$ if $V' \subseteq V$ and there is a injection $\phi : E \to E'$ where $\phi(e) \subseteq e$. Thus a subhypergraph of H is obtained be deleting vertices and edges and trimming edges[4].

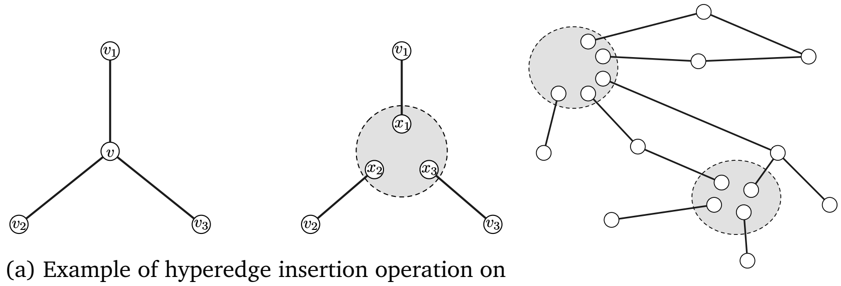
For simplicity, given hypergraph $H = (V, E)$, we use n as the number of vertices, m as the number of edges, and $p = \sum_{e \in E} |e|$ as the sum of degrees.

Equivalent digraph: s-t mincut in a hypergraph H can be computed via an s-t maximum flow in an associated capacitated digraph (see [27]) $\vec{H} = (\vec{V}, \vec{E})$ that we call the equivalent digraph. $\vec{H} = (\vec{V}, \vec{E})$ is defined as follows:

1. $\vec{V} = V \cup E^+ \cup E^-$, where $E^+ = \{e^+ | e \in E\}$ and $E^- = \{e^- | e \in E\}$.
2. If $v \in e$ for $v \in V$ and $e \in E$ then (v, e^-) and (e^+, v) are in $\vec{E}$ with infinite capacity.
3. For each $e \in E$, $(e^-, e^+) \in \vec{E}$ has capacity equal to $c(e)$.

For any pair $s, t \in V(H)$, there is bijection between the finite capacity s-t cuts in $\vec{H}$ and s-t cuts in H. We omit further details of this simple fact.

[4]The standard definition of subhypergraph does not allow trimming, but our definition is natural for sparsification.



(a) Example of hyperedge insertion operation on vertex v

(b) A hypercactus

Figure 2.1: Examples of hypercactus. Grey regions are hyperedges.

Cactus and hypercactus: A *cactus* is a graph in which every edge is in at most one cycle. A *hypercactus* is a hypergraph obtained by a sequence of *hyperedge insertions* starting from a cactus. A hyperedge insertion is defined as follows. A vertex v in a hypergraph with degree at least 3 and only incident to edges of rank 2, say $vv_1, \ldots, vv_k$ is called a v-star. A hyperedge insertion replaces a v-star by deleting v, adding new vertices $x_1, x_2, \ldots, x_k$, adding new edges $\{x_1, v_1\}, \{x_2, v_2\}, \ldots, \{x_k, v_k\}$ and a new hyperedge $\{x_1, x_2, \ldots, x_k\}$. See Figure 2.1 for examples.

2.1 Vertex orderings for hypergraphs

We work with several vertex orderings defined for hypergraphs. Given a hypergraph $H = (V, E)$ and an ordering of the vertices $v_1, \ldots, v_n$, several other orderings and quantities are induced by such an ordering. The *head* of an edge $h(e)$, defined as $v_{\min\{j|v_j \in e\}}$, is the first vertex of e in the ordering. An ordering of the edges $e_1, \ldots, e_m$ is called *head ordering*, if $\min\left\{j|v_j \in e_i\right\} \leq \min\{j|v_j \in e_{i+1}\}$. An edge e is a *backward edge* of v if $v \in e$ and $h(e) \neq v$. The head of a backward edge incident to v comes before v in the vertex order. $V_i = \{v_1, \ldots, v_i\}$ are the first i vertices in the ordering.

A pair of vertices (s, t) is called a *pendant pair* if $\{t\}$ is a minimum s-t cut. There are three algorithms for computing a hypergraph mincut following the Nagamochi-Ibaraki approach of finding a pendant pair, contracting the pair, and recursing. All three algorithms find a pendant pair by computing an ordering of the vertices and showing that the last two vertices form a pendant pair. We describe these orderings below.

Definition An ordering of vertices $v_1, \ldots, v_n$ is called

1. a *maximum adjacency ordering* or *MA-ordering* if for all $1 \leq i < j \leq n$, $d(V_{i-1}, v_i) \geq d(V_{i-1}, v_j)$.

2. a *tight ordering* if for all $1 \leq i < j \leq n$, $d'(V_{i-1}, v_i) \geq d'(V_{i-1}, v_j)$.

3. a *Queyranne ordering* if for all $1 \leq i < j \leq n$, $d(V_{i-1}, v_i) + d'(V_{i-1}, v_i) \geq d(V_{i-1}, v_j) + d'(V_{i-1}, v_j)$.

In graphs the three orderings coincide if the starting vertex is the same and ties are broken in the same way. However, they can be different in hypergraphs. As an example, consider a hypergraph with vertices a, x, y, z and four edges with capacities as follows: $c(\{a, x\}) = 4, c(\{a, y\}) = 3, c(\{a, x, z\}) = 4$ and $c(\{a, y, z\}) = 8$. Capacities can be avoided by creating multiple copies of an edge. Consider orderings starting with a. It can be verified that the second vertex has to be x, y and z for tight, Queyranne, and MA-ordering respectively which shows that they have to be distinct.

Klimmek and Wagner used the MA-ordering [25]. Mak and Wong used the tight ordering [30]. Queyranne defined an ordering for symmetric submodular functions [36] which when specialized to cut functions of hypergraphs is the one we define; we omit a formal proof of this observation. All three orderings can be computed in $O(p + n \log n)$ time for capacitated hypergraphs, and in $O(p)$ time for uncapacitated hypergraphs. We do not use the Queyranne ordering in this paper but rely on the other two. We state a lemma that summarizes the crucial property that the three orderings share regarding the mincut between the last two vertices in the ordering.

Lemma 2.1 *Let $v_1, \ldots, v_n$ be a MA-ordering, a tight ordering or a Queyranne ordering of a hypergraph, then $\{v_n\}$ is a v_{n-1}-v_n mincut and $\lambda(v_{n-1}, v_n) = d'(V_{n-1}, v_n) = d(V_{n-1}, v_n)$.*

A stronger property holds for the MA-ordering.

Lemma 2.2 *Let $v_1, \ldots, v_n$ be an MA-ordering of hypergraph H, then $\lambda(v_{i-1}, v_i) \geq d(V_{i-1}, v_i)$ for all $1 < i \leq n$.*

Proof: Let $H_i = (V_i, E_i)$, where $E_i = \{e \cap V_i | e \in E\}$. One can check that $v_1, \ldots, v_i$ is an MA-ordering of H_i, and hence,

$$\lambda(v_{i-1}, v_i; H) \geq \lambda(v_{i-1}, v_i; H_i) = d(V_{i-1}, v_i; H_i) = d(V_{i-1}, v_i; H).$$

□

3 Sparsification and faster mincut algorithm for small λ

This section shows that a well-known sparsification result for graphs can be generalized to hypergraphs. The hypergraphs in this section are uncapacitated.

Given an uncapacitated hypergraph H and a non-negative integer k, the goal of sparsification is to find a sparse subhypergraph H' of H such that $\lambda_{H'}(s,t) \geq \min(k, \lambda_H(s,t))$ for all $s, t \in V$. We call such a subhypergraph a k-sparsifier. It is known that there exists a subhypergraph with $O(kn)$ edges through edge deletions [19]. The fact that such a sparsifier exists is not hard to prove. One can generalize the forest decomposition technique for graphs [33] in a straight-forward way. However, the sum of degrees of the resulting sparsifier could still be large. Indeed, there might not exist any k-sparsifier with sum of degree $O(kn)$ through edge deletion alone. Consider the following example. Let $H = (V, E)$ be a hypergraph on n vertices and $n/2 - 1$ edges (assume n is even) where $E = \left\{e_1, \ldots, e_{n/2-1}\right\}$ and $e_i = \left\{v_i, v_{n/2}, \ldots, v_n\right\}$ for $1 \leq i < n/2$. Any connected subhypergraph of H has to contain all the edges and thus, even for $k = 1$, the sum of degrees is $\Omega(n^2)$.

However, if trimming of edges is allowed, we can prove the following stronger result.

Theorem 3.1 *Let $H = (V, E)$ be a hypergraph on n vertices and m edges with $\mathsf{sum\text{-}deg}(H) = p$. There is a data structure that can be created in $O(p)$ time, such that for any given non-negative integer k, it can return a k-sparsifier H' of H in $O(\mathsf{sum\text{-}deg}(H'))$ time with the property that $\mathsf{sum\text{-}deg}(H') = O(kn)$.*

Our proof is an adaptation of that of Frank, Ibaraki and Nagamochi [13] for the graph version of the sparsification theorem.

Given a hypergraph $H = (V, E)$ consider an MA-ordering $v_1, \ldots, v_n$ and let $e_1, \ldots, e_m$ be the induced head ordering of the edges. Let $D_k(v)$ to be the first k backward edges of v in the head ordering, or all the backward edges of v if there are fewer than k backward edges. For each vertex v and backward edge e of v, we remove v from e if $e \notin D_k(v)$. The new hypergraph from this operation is H_k. Formally, given H and k, $H_k = (V, E_k)$ is defined as follows. For an edge $e \in E$ let e' denote the edge $\{v \mid v \in e \in D_k(v) \text{ or } v = h(e)\}$; E_k is defined to be the edge set $\{e' \mid e \in E, |e'| \geq 2\}$. It is easy to see that if $j \leq k$, H_j is a subhypergraph of H_k.

We observe that $\mathsf{sum\text{-}deg}(H_k) \leq 2kn$. Each vertex v is in at most k backward edges in H_k for a total contribution of at most kn to the sum of degrees, and the remaining contribution of at most kn comes from head of each edge which can be charged to the backward edges.

We sketch a data structure that can be created from hypergraph H in $O(p)$ time, such that for all k, the data structure can retrieve H_k in $O(kn)$ time. First, we compute the MA-ordering, which takes $O(p)$ time. Using the MA-ordering, we obtain the induced head ordering of the edges, and the head for each edge, again in $O(p)$ time; we omit the simple details for this step. For each vertex v, we can sort all the backedges of v use the head ordering in $O(p)$ time as follows: we maintain a queue Q_v for each vertex v, and inspect the edges one by one in the head ordering. When e is inspected, we push e into queue Q_v if $v \in e$ and v is not the head of e. This completes the preprocessing phase for the data structure. To retrieve H_k, we maintain a set of queues that eventually represent the set of edges E_k. For each vertex v, find the edges in $D_k(v)$, which is exactly the first k edges (or all edges if there are fewer than k edges) in Q_v. For each edge $e \in D_k(v)$, we push v into a queue Q_e (if Q_e was not already created, we first create Q_e and push the head vertex of e into Q_e). At the end of the process, each queue Q_e contains the vertices of an edge in E_k. The running time is $O(\mathsf{sum\text{-}deg}(H_k)) = O(kn)$ since we only process $D_k(v)$ for each v.

It remains to show that H_k is a k-sparsifier of H. In fact, we will show H_k preserves more than just connectivity, it also preserves all cuts up to value k. Namely $|\delta_{H_k}(A)| \geq \min(k, |\delta_H(A)|)$ for all $A \subseteq V$.

Lemma 3.2 *If $v_1, \ldots, v_n$ is an MA-ordering of H, and H_k is a hypergraph obtained from H via the ordering, then $v_1, \ldots, v_n$ is an MA-ordering of H_k.*

Proof: By construction, $d(V_i, v_j; H_k) \leq \min(k, d(V_i, v_j; H))$ for all $i < j$. Consider the first $\min\{k, d(V_i, v_j; H)\}$ edges incident to v_j in the head ordering; v_j is not trimmed from them. Hence $d(V_i, v_j; H_k) \geq \min\{k, d(V_i, v_j; H)\}$.

For all $i \le j$,

$$d(V_{i-1}, v_i; H_k) \ge \min\{k, d(V_{i-1}, v_i; H)\} \ge \min\{k, d(V_{i-1}, v_j; H)\} \ge d(V_{i-1}, v_j; H_k).$$

This establishes that $v_1, \ldots, v_n$ is an MA-ordering for H_k. □

For $X \subseteq V$ we define $\gamma(X) = \{e \mid e \cap X \neq \emptyset\}$ to be the set of edges that contain at least one vertex from X. We need a helper lemma below.

Lemma 3.3 *Let $H = (V, E)$ be a hypergraph and $A, B \subseteq V$. For $u \in B$ and $v \in V$, if $E(u, v) \subseteq \delta(A) \cap \gamma(B)$, then*

$$|\gamma(B) \cap \delta(A)| \ge |\gamma(B-u) \cap \delta(A)| + |E(u, v) \setminus E(B-u, u, v)|.$$

Proof: Consider an edge $e \in E(u, v) \setminus E(B-u, u, v)$. We claim that $e \notin \gamma(B-u)$. Indeed, if $e \in \gamma(B-u)$, then e is an edge that intersects $B-u$, $\{u\}$ and $\{v\}$, but then $e \in E(B-u, u, v)$. This shows $E(u, v) \setminus E(B-u, u, v)$ is disjoint from $\gamma(B-u)$, and therefore disjoint from $\gamma(B-u) \cap \delta(A)$.

We have (i) $\gamma(B-u) \cap \delta(A) \subseteq \gamma(B) \cap \delta(A)$ since $\gamma(B-u) \subseteq \gamma(B)$, and (ii) $E(u, v) \setminus E(B-u, u, v) \subseteq \gamma(B) \cap \delta(A)$ by assumption. Since we have argued that $\gamma(B-u)$ and $E(u, v) \setminus E(B-u, u, v)$ are disjoint, we have the desired inequality

$$|\gamma(B) \cap \delta(A)| \ge |\gamma(B-u) \cap \delta(A)| + |E(u, v) \setminus E(B-u, u, v)|.$$

□

Lemma 3.4 *Let $v_1, \ldots, v_n$ be an MA-ordering for $H = (V, E)$. Then, for all $i < j$ and $A \subseteq V$ such that $v_i \in A$ and $v_j \notin A$. $|\gamma(V_{i-1}) \cap \delta(A)| \ge d(V_{i-1}, v_j)$.*

Proof: Proof by induction on (i, j) ordered lexicographically. For the base case consider $i = 1$ and $j > 1$. Indeed, in this case both sides of the inequality are 0 and the desired inequality holds trivially. Assume lemma is true for all (i', j') where $1 \le i' < j'$, such that $j' < j$ or $j' = j$ and $i' < i$. We consider two cases.

Case 1: $v_{i-1} \in A$. Then because $v_j \notin A$, $E(v_{i-1}, v_j) \subseteq \delta(A) \cap \gamma(V_{i-1})$. We apply Lemma 3.3 with $B = V_{i-1}$ and $u = v_{i-1}$ and $v = v_j$ to obtain:

$$\begin{aligned} |\gamma(V_{i-1}) \cap \delta(A)| &\ge |\gamma(V_{i-2}) \cap \delta(A)| + |E(v_{i-1}, v_j) \setminus E(V_{i-2}, v_{i-1}, v_j)| \\ &\ge d(V_{i-2}, v_j) + |E(v_{i-1}, v_j) \setminus E(V_{i-2}, v_{i-1}, v_j)| \quad \text{(apply induction on } (i-1, j), A) \\ &= d(V_{i-1}, v_j). \end{aligned}$$

Case 2: $v_{i-1} \notin A$. Note that $i \ge 2$. Consider $A' = V \setminus A$. We have $v_{i-1} \in A'$ and $v_i \notin A'$; therefore, $E(v_{i-1}, v_i) \subseteq \delta(A') \cap \gamma(V_{i-1})$. Applying Lemma 3.3 with $B = V_{i-1}$, $u = v_{i-1}$ and $v = v_i$,

$$\begin{aligned} |\gamma(V_{i-1}) \cap \delta(A')| &\ge |\gamma(V_{i-2}) \cap \delta(A')| + |E(v_{i-1}, v_i) \setminus E(V_{i-2}, v_{i-1}, v_i)| \\ &\ge d(V_{i-2}, v_i) + |E(v_{i-1}, v_i) \setminus E(V_{i-2}, v_{i-1}, v_i)| \quad \text{(apply induction on } (i-1, i), A') \\ &= d(V_{i-1}, v_i) \\ &\ge d(V_{i-1}, v_j) \quad \text{(from MA-ordering)}. \end{aligned}$$

Since $\delta(A') = \delta(V \setminus A) = \delta(A)$, $|\gamma(V_{i-1}) \cap \delta(A')| = |\gamma(V_{i-1}) \cap \delta(A)|$.

This finishes the proof. □

Using the preceding lemma we finish the proof that H_k is a k-sparsifier.

Theorem 3.5 *For every $A \subseteq V$, $|\delta_{H_k}(A)| \ge \min(k, |\delta_H(A)|)$.*

Proof: By induction on k. The statement is clearly true for $k = 0$. $|\delta_{H_i}(A)| \le |\delta_{H_k}(A)|$ for all $i \le k$, because H_i is a subhypergraph of H_k.

Consider any $k > 0$. If $|\delta_H(A)| = k' < k$, then by induction,

$$k' = |\delta_{H_{k'}}(A)| \le |\delta_{H_k}(A)| \le |\delta_H(A)| = k'.$$

Therefore, it suffices to only consider A such that $|\delta_H(A)| \geq k$. We will derive a contradiction assuming that $|\delta_{H_k}(A)| < k$. Since $|\delta_H(A)| \geq k$, there exists an edge $e \in E(H)$ such that $e \in \delta_H(A)$ but was either trimmed to $e' \in E(H_k)$ such that $e' \notin \delta_{H_k}(A)$ or e is removed completely because it is trimmed to be a singleton $\{h(e)\}$. Let $v_i = h(e)$ and without loss of generality we can assume $v_i \in A$ (otherwise we can consider $\bar{A}$). Since e' does not cross A in H_k, there is a $v_j \in e \cap \bar{A}$ with $j > i$ (since v_i is the head of e) and v_j was trimmed from e during the sparsification.

$D_k(v_j)$ has exactly k edges because backward edge e of v_j is not in $D_k(v_j)$. For each $f \in D_k(v_j)$, the trimmed $f' \in E(H_k)$ contains both $h(f) = v_\ell$ and v_j; we claim that for each such f, $\ell \leq i$ for otherwise e would be ahead of f in the head order and v_j would be trimmed from f before it is trimmed from e. From this we obtain that $E(V_{j-1}, v_j; H_k) = E(V_i, v_j; H_k)$ and hence $d(V_{j-1}, v_j; H_k) = d(V_i, v_j; H_k) = k$.

For the remainder of the the proof, we only work with H_k and the quantities d, δ, E, γ are with respect to this hypergraph and not H. We have

$$k = d(V_{j-1}, v_j) = d(V_i, v_j) = d(V_{i-1}, v_j) + d(v_i, v_j) - d(V_{i-1}, v_i, v_j).$$

Hence $d(V_{i-1}, v_j) = k - d(v_i, v_j) + d(V_{i-1}, v_i, v_j)$.

From Lemma 3.2 $v_1, \ldots, v_n$ is an MA-ordering of H_k as well. Applying Lemma 3.4 to H_k, v_i and v_j and A, $|\gamma(V_{i-1}) \cap \delta(A)| \geq d(V_{i-1}, v_j)$. Combining this inequality with the preceding one,

$$|\gamma(V_{i-1}) \cap \delta(A)| \geq d(V_{i-1}, v_j) = k - d(v_i, v_j) + d(V_{i-1}, v_i, v_j). \tag{1}$$

We also observe that $E(V_{i-1}, v_i, v_j) \subseteq E(v_i, v_j) \subseteq \delta(A)$ because $v_i \in A$ and $v_j \notin A$. We obtain a contradiction by the following set of inequalities:

$$\begin{aligned}
k &> |\delta(A)| \quad \text{(by assumption)} \\
&\geq |\gamma(V_i) \cap \delta(A)| \\
&= |\gamma(V_{i-1}) \cap \delta(A)| + |E(v_i, v_j) \setminus E(V_{i-1}, v_i, v_j)| \\
&= |\gamma(V_{i-1}) \cap \delta(A)| + d(v_i, v_j) - d(V_{i-1}, v_i, v_j) \\
&\geq \big(k - d(v_i, v_j) + d(V_{i-1}, v_i, v_j)\big) + d(v_i, v_j) - d(V_{i-1}, v_i, v_j) \quad \text{(from (1))} \\
&= k
\end{aligned}$$

This finishes the proof. □

There are applications where one want a k-sparsifier with $O(kn)$ edges using only deletion. This can be done easily by first compute H_k, and for each edge in H_k, replace it with the original edge in H.

One can ask whether tight ordering or Queyranne ordering would also lead to sparsification. We observe that they do not work if the only modification is the input ordering, and H_k is constructed the same way through the head ordering of the edges.

For tight ordering, consider $H = (\{0,1,2,3\}, E)$, where $E = \{\{0,1,2\}, \{0,2,3\}, \{1,2\}\}$. $0,1,2,3$ is a tight ordering. H_2 is H with edge $\{1,2\}$ removed. $\lambda_{H_2}(1,2) = 1 < 2 = \lambda_H(1,2)$.

For Queyranne ordering, consider $H = (\{0,\ldots,4\}, E)$, where

$$E = \{\{0,1,2\}, \{0,1,2,3\}, \{0,1,3,4\}, \{1,3,4\}, \{2,3\}\}.$$

$0,1,2,3,4$ is a Queyranne ordering. H_3 is all the edges except the edge $\{2,3\}$. We have $\lambda_{H_3}(2,3) = 2 < 3 = \lambda_H(2,3)$.

There may be other ways to obtain a sparsifier using these orderings but we have not explored this.

Algorithmic applications: Computing connectivity in uncapacitated hypergraphs can be sped up by first sparsifying the given hypergraph and then running a standard algorithm on the sparsifier. This is especially useful when one is interested in small values of connectivity. For global mincut we obtain the following theorem.

Theorem 3.6 *The global mincut of a uncapacitated hypergraph H with n vertices and $\mathsf{sum\text{-}deg}(H) = p$ can be computed in $O(p + \lambda n^2)$ time, where λ is the value of the mincut of H.*

Proof: Using Theorem 3.1, we first compute a data structure in $O(p)$ time that allows us to retrieve H_k in $O(kn)$ time for any given k. Suppose we knew a number k such that $k \geq \lambda$ and $k = O(\lambda)$. We can compute the k-sparsifier H_k in $O(kn)$ time and compute $\lambda(H_k) = \lambda(H)$ in $O(\lambda n^2)$ time using one of the known algorithms since $\mathsf{sum\text{-}deg}(H_k) = O(\lambda n)$. To find k we apply exponential search for the smallest i such that $2^i > \lambda$. Each search computes hypergraph mincut on H_{2^i}, which takes $O(2^i n^2)$ time. For any $k > \lambda$, the value of the mincut on the k-sparsifier equals to λ. Therefore, the search stops when the value of mincut of H_{2^i} is strictly smaller than 2^i. The total time for the computation is $O(p + \sum_{i=1}^{1+\lceil \log \lambda \rceil} 2^i n^2) = O(p + \lambda n^2)$. □

A similar idea can be used to compute the edge-connectivity in H between some given pair s, t. An algorithm for s-t connectivity that runs in time $T(n, m, p)$ on a hypergraph with n nodes, m edges and sum of degrees p can be sped up to $T(n, m, \lambda(s, t)n)$. Sparsification can also help in computing α-approximate mincuts for $\alpha > 1$ via the stronger property guaranteed by Theorem 3.5.

4 Canonical decomposition and Hypercactus Representation

In this section, we are interested in finding a canonical decomposition of a capacitated hypergraph which captures, in a compact way, information on all the mincuts. Cunningham [7] proved that such decomposition exists for an arbitrary non-negative submodular function, following previous work by Cunningham and Edmonds [8] and Fujishige [15]. Cheng [6] showed that the canonical decomposition can be used to efficiently and relatively easily build a hypercactus representation, and later Fleiner and Jordan [10] showed a similar result for arbitrary symmetric submodular functions. In a sense, one can view the canonical decomposition as a more fundamental object since it has uniqueness properties while cactus and hypercactus representations are not necessarily unique.

As noted already by Cunningham, the key tool needed to build a canonical decomposition is an algorithm to find a non-trivial mincut. Here we show an efficient algorithm for finding such a mincut in a hypergraph, and then use it to build a canonical decomposition. We can then construct a hypercactus from the canonical decomposition as shown in [6]. We believe that this approach is easier to understand and conceptually simpler than the existing deterministic cactus construction algorithms for graphs that build the cactus directly.

A cut is *trivial* if one side of the cut has exactly one vertex. A *split* is a non-trivial mincut. An s-t split is a split that separates s and t.

4.1 An efficient split oracle for hypergraphs

Given a hypergraph H we would like to find a split if one exists. It is not hard to come up with a polynomial-time algorithm for this task but here we wish to design a faster algorithm. We accomplish this by considering a weaker guarantee which suffices for our purposes. The algorithm, given H and the mincut value λ, outputs either a split in H or a pair of vertices $\{s, t\}$ such that there is no s-t split in H. We call such an algorithm a split oracle. We describe a near-linear-time split oracle.

We first show how to use a maximum s-t flow in $\vec{H}$ to help decide whether there is an s-t split, and compute one if there is.

Lemma 4.1 *Given a maximum s-t flow in the equivalent digraph of H, and the value of mincut λ in H, there is an algorithm that in $O(p)$ time either finds a s-t split, or certifies that no s-t split exists in H.*

Proof: If the value of the maximum s-t flow is greater than λ, there is no s-t split. Otherwise, there is an s-t split iff there is a non-trivial min-s-t cut in H.

Suppose a directed graph G has k minimum u-v-cuts for some vertex pair (u, v). Given a maximum u-v flow in G and an integer ℓ, there is an enumeration algorithm [35] that outputs $\min(\ell, k)$ distinct min-u-v-cuts in $O(\ell m)$ time where m is the number of edges in G.

We run the enumeration algorithm with $\ell = 3$ on $\vec{H}$ for the pair (s, t). Every min-s-t cut in $\vec{H}$ corresponds to a min-s-t cut in H. If the algorithm returns at most two cuts and both are trivial then there is no s-t split. Otherwise one of the output cuts is an s-t split. The running time is $O(p)$ since the number of edges in $\vec{H}$ is $O(p)$. □

One can find a maximum s-t flow in $\vec{H}$ using standard flow algorithms but that would not lead to a near-linear time algorithm. In graphs, Arikati and Mehlhorn [2] devised a linear-time algorithm that computes the maximum flow between the last two vertices of an MA-ordering. Thus, we have a near-linear-time split oracle for graphs. Recall that in hypergraphs there are three orderings which all yield a pendant pair. We generalized Arikati and Mehlhorn's algorithm to a linear-time algorithm that tries to find a maximum flow between the last two vertices of an MA-ordering of a hypergraph (the flow is in the equivalent digraph). Even though it appears to correctly compute a maximum flow in all the experiments we ran, we could not prove its correctness. Instead we found a different method based on the tight ordering, that we describe below.

Let $v_1, v_2, \ldots, v_n$ be a tight ordering for a hypergraph $H = (V, E)$. We define a tight graph $G = (V, E')$ with respect to H and the given tight ordering as follows. For each edge $e \in E$, we add an edge e' to E', where e' consists of the last 2 vertices of e under the tight ordering. The key observation is the following.

Lemma 4.2 *Suppose $H = (V, E)$ is a hypergraph and $v_1, \ldots, v_n$ is a tight ordering for H, and $G = (V, E')$ is the corresponding tight graph. Then, for $1 \le i < j \le n$, $d'(V_i, v_j; G) = d'(V_i, v_j; H)$.*

Proof: Consider any edge e counted in $d'(V_i, v_j; H)$. $e \subseteq V_i \cup \{v_j\}$, and e contains v_j. e' contains v_j, and the other end of e' is in V_i. Therefore e' is counted in $d'(V_i, v_j; G)$. This shows that $d'(V_i, v_j; H) \le d'(V_i, v_j; G)$.

To see the other direction, consider an $e' \in E$ corresponding to an edge $e \in E$. If e' is counted in $d'(V_i, v_j; G)$, it must be that v_j is the last vertex in e and the second last vertex of e is in V_i. This implies that $e \subseteq V_i \cup \{v_j\}$, and therefore counted in $d'(V_i, v_j; H)$, and completes the direction $d'(V_i, v_j; H) \ge d'(V_i, v_j; G)$. □

The preceding lemma implies that the tight ordering for H is a tight ordering for G. From Lemma 2.1,

$$\lambda(v_{n-1}, v_n; G) = d'(V_{n-1}, v_n; G) = d'(V_{n-1}, v_n; H) = \lambda(v_{n-1}, v_n; H)$$

Letting $s = v_{n-1}$ and $t = v_n$, we see that $\lambda(s, t; G) = \lambda(s, t; H)$. Moreover, an s-t flow in G can be easily lifted to an s-t flow in $\vec{H}$. Thus, we can compute an s-t max flow in G in linear-time using the algorithm of [2] and this can be converted, in linear time, into an s-t max-flow in $\vec{H}$.

This gives the following theorem.

Theorem 4.3 *The split oracle can be implemented in $O(p + n \log n)$ time for capacitated hypergraphs, and in $O(p)$ time for uncapacitated hypergraphs.*

4.2 Decompositions, Canonical and Prime

We define the notion of decompositions to state the relevant theorem on the existence of a canonical decomposition. In later subsections we describe the computational aspects.

A hypergraph H is *prime* if it does not contain any split; in other words all mincuts of H are trivial. A capacitated hypergraph is called a *solid polygon* if it consists of a cycle where each edge has the same capacity a and a hyperedge containing all vertices with capacity b. If $a = 0$, it is called *brittle*, otherwise it is called *semi-brittle*. A solid polygon is *not* prime if it has at least 4 vertices. For a semi-brittle hypergraph with at least 4 vertices, every split consists of two edges on the cycle and the hyperedge covering all vertices. For a brittle hypergraph with at least 4 vertices, any non-trivial cut is a split.

Given a hypergraph $H = (V, E)$ and a set U, a function $\phi : V \to U$ defines a new hypergraph through a sequence of contraction operations as follows: for each element $u \in U$, contract $\phi^{-1}(u)$ into u. The resulting hypergraph is the ϕ-contraction of H. A hypergraph obtained from $H = (V, E)$ by contracting $S \subseteq V$ into a single vertex is denoted by H/S.

$\{H_1, H_2\}$ is a *simple refinement* of H if H_1 and H_2 are hypergraphs obtained through a split (V_1, V_2) of H and a new *marker* vertex x as follows.

1. H_1 is H/V_2, such that V_2 gets contracted to x.

2. H_2 is H/V_1, such that V_1 gets contracted to x.

If $\{H_1, H_2\}$ is a simple refinement of H, then mincut value of H_1, H_2 and H are all equal.

A set of hypergraphs $\mathcal{D} = \{H_1, H_2, \ldots, H_k\}$ is called a *decomposition* of a hypergraph H if it is obtained from $\{H\}$ by a sequence of operations each of which consists of replacing one of the hypergraphs in the set by its simple refinement; here we assume that each operation uses new marker vertices. A decomposition $\mathcal{D}$ is a simple refinement of decomposition $\mathcal{D}'$ if $\mathcal{D}$ is obtained through replacing one of the hypergraph in $\mathcal{D}'$ by its simple refinement. A decomposition $\mathcal{D}'$ is a *refinement* of $\mathcal{D}$ if $\mathcal{D}'$ is obtained through a sequence of simple refinement operations from $\mathcal{D}$. If the sequence is non-empty, $\mathcal{D}'$ is called a *strict refinement*. Two decompositions are *equivalent* if they are the same up to relabeling of the marker vertices. A decomposition is *minimal* with property $\mathcal{P}$ if it is not a strict refinement of some other decomposition with the same property $\mathcal{P}$. A *prime* decomposition is one in which all members are prime. A decomposition is *standard* if every element is either prime or a solid polygon.

Every element in the decomposition is obtained from a sequence of contractions from H. Hence we can associate each element H_i in the decomposition with a function $\phi_{H_i} : V \to V(H_i)$, such that H_i is the ϕ_{H_i}-contraction of H. Every decomposition $\mathcal{D}$ has an associated *decomposition tree* obtained by having a node for each hypergraph in the decomposition and an edge connecting two hypergraphs if they share a marker vertex.

The important theorem below is due to [7], and stated again in [6] specifically for hypergraphs.

Theorem 4.4 ([7]) *Every hypergraph H has a unique (up to equivalence) minimal standard decomposition. That is, any two minimal standard decompositions of H differ only in the labels of the marker vertices.*

The unique minimal standard decomposition is called the *canonical decomposition*. As a consequence, *every* standard decomposition is a refinement of the canonical decomposition. We remark that minimality is important here. It captures all the mincut information in H as stated below.

Theorem 4.5 ([6, 7]) *Let $\mathcal{D} = \{H_1, \ldots, H_k\}$ be a canonical decomposition of H.*

1. *For each mincut S of H, there is a unique i, such that $\phi_{H_i}(S)$ is a mincut of H_i.*

2. *For each mincut S of H_i, $\phi_{H_i}^{-1}(S)$ is a mincut of H.*

Note that each hypergraph in a canonical decomposition is either prime or a solid polygon and hence it is easy to find all the mincuts in each of them. We observe that any decomposition $\mathcal{D}$ of H can be compactly represented in $O(n)$ space by simply storing the vertex sets of the hypergraph in $\mathcal{D}$.

Recall that a set of edges $E' \subseteq E$ is called a min edge-cut-set if $E' = \delta(S)$ for some mincut S. As a corollary of the preceding theorem, one can easily prove that there are at most $\binom{n}{2}$ distinct min edge-cut-sets in a hypergraph. This fact is not explicitly stated in [6, 7] or elsewhere in the literature but was known to those familiar with the decomposition theorem.

Corollary 4.6 *A hypergraph with n vertices has at most $\binom{n}{2}$ distinct min edge-cut-sets.*

Proof: Let H be a hypergraph on n vertices. If H is prime, then there are at most n min edge-cut-sets. If H is a solid-polygon, then there are at most $\binom{n}{2}$ min edge-cut-sets. Let $\mathcal{D}$ be a canonical decomposition of H. $\mathcal{D}$ is obtained via a simple refinement $\{H_1, H_2\}$ of H with size a and b, followed by further refinement. Then, by induction, there are at most $\binom{a}{2}+\binom{b}{2}$ min edge-cut-sets in H_1 and H_2. Here $a+b = n+2$ and $a, b \leq n-1$. Therefore $\binom{a}{2} + \binom{b}{2} \leq \binom{3}{2} + \binom{n-1}{2} \leq \binom{n}{2}$ when $n \geq 4$. □

4.3 Computing a canonical decomposition

In this section we describe an efficient algorithm for computing the canonical decomposition of a hypergraph H.

We say that two distinct splits $(A, \bar{A})$ and $(B, \bar{B})$ cross if A and B cross, otherwise they do not cross. One can easily show that every decomposition is equivalently characterized by the set of non-crossing splits induced by the marker vertices. Viewing a decomposition as a collection of non-crossing splits is convenient since it does not impose an order in which the splits are processed to arrive at the decomposition — any ordering of processing the non-crossing splits will generate the same decomposition.

Call a split *good* if it is a split that is not crossed by any other split; otherwise the split is called bad. A canonical decomposition corresponds to the collection of all good splits. The canonical decomposition can be obtained through the set of of good splits [8, Theorem 3] via the following simple algorithm. If H is prime or solid polygon return $\{H\}$ itself. Otherwise find a good split $(A, \bar{A})$ and the simple refinement $\{H_1, H_2\}$ of H induced by the split and return the union of the canonical decompositions of H_1 and H_2 computed recursively. Unfortunately, finding a good split directly is computationally intensive.

On the other hand finding a prime decomposition can be done via a split oracle by a simple recursive algorithm, as we shall see in Section 4.4. Note that a prime decomposition is not necessarily unique. We will build a canonical decomposition through a prime decomposition. This was hinted in [7], but without details and analysis. Here we formally describe such an algorithm.

One can go from a prime decomposition to a canonical decomposition by removing some splits. Removing a split corresponds to gluing two hypergraphs with the same marker vertex into another hypergraph resulting in a new decomposition. We formally define the operation as follows. Suppose $\mathcal{D}$ is a decomposition of H with a marker vertex x contained in H_1 and H_2. We define a new contraction of H obtained by *gluing* H_1 and H_2. Let ϕ_{H_1} and ϕ_{H_2} be the contractions of H, respectively. Define function $\phi' : V \to (V(H_1) \cup V(H_2)) - x$ as follows

$$\phi'(v) = \begin{cases} \phi_{H_1}(v) & \text{if } \phi_{H_2}(v) = x \\ \phi_{H_2}(v) & \text{if } \phi_{H_1}(v) = x \end{cases}$$

H_x is the contraction of H defined by ϕ'. The gluing of $\mathcal{D}$ through x is the set $\mathcal{D}_x = \mathcal{D} - \{H_1, H_2\} \cup \{H_x\}$. The operation reflects removing the split induced by x from the splits induced by $\mathcal{D}$, therefore it immediately implies the following lemma.

Lemma 4.7 *$\mathcal{D}_x$ is a decomposition of H. Moreover, $\mathcal{D}_x$ can be computed from $\mathcal{D}$ and H in $O(p)$ time.*

Remark In order to compute D_x implicitly, we only have to obtain ϕ_{H_1}, ϕ_{H_2} and compute a single contraction. Therefore $O(p)$ space is sufficient if we can obtain ϕ_{H_1} and ϕ_{H_2} in $O(p)$ time and space.

We need the following simple lemma.

Lemma 4.8 *Let H be a solid polygon. Any decomposition of H is a standard decomposition. Thus, if $\mathcal{D}$ is a decomposition of H, for any marker vertex, gluing it results in a standard decomposition of H.*

Proof: We first prove that any decomposition of H is a standard decomposition. This is by induction. If the solid polygon consists of a cycle with positive capacity, then exactly two edges in the cycle and the edge that contains all vertices crosses a split. One can verify that contraction of either side of the split results in a solid polygon or a prime hypergraph. Otherwise, the solid polygon is a single hyperedge covering all vertices. Any contraction of this hypergraph is a solid polygon.

The second part of the lemma follows from the first and the fact that gluing results in a decomposition. □

The following lemma is easy to see.

Lemma 4.9 *Given a hypergraph H there is an algorithm to check if H is a solid polygon in $O(p)$ time.*

Adding a split corresponds to a simple refinement. Therefore a decomposition $\mathcal{D}'$ is a refinement of $\mathcal{D}$ then the set of induced splits of $\mathcal{D}$ is a subset of induced splits of $\mathcal{D}'$.

Consider the following algorithm that starts with a prime decomposition $\mathcal{D}$. For each marker x, inspect if gluing through x results in a standard decomposition; one can easily check via the preceding lemma whether the gluing results in a solid polygon which is the only thing to verify. If it is, apply the gluing, if not, move on to the next marker. Every marker will be inspected at most once, therefore the algorithm stops after $O(n)$ gluing operations and takes time $O(np)$. Our goal is to show the correctness of this simple algorithm.

The algorithm starts with a prime decomposition $\mathcal{D}$ which is a standard decomposition. If it is minimal then it is canonical and no gluing can be done by the algorithm (otherwise it would violate minimality) and we will output a canonical decomposition as required. If $\mathcal{D}$ is not minimal then there is a canonical decomposition $\mathcal{D}^*$ such that $\mathcal{D}$ is a strict refinement of $\mathcal{D}^*$. Let $\mathcal{D}^* = \{H_1, H_2, \ldots, H_k\}$ where each H_i is prime or a solid polygon. Therefore $\mathcal{D} = \cup_{i=1}^{k} \mathcal{D}_i$ where $\mathcal{D}_i$ is a refinement of H_i. If H_i is prime than $\mathcal{D}_i = \{H_i\}$. If H_i is a solid polygon then $\mathcal{D}_i$ is a standard decomposition of H_i. Our goal is to show that irrespective of the order in which we process the markers in the algorithm, the output will be $\mathcal{D}^*$. Let the marker set for $\mathcal{D}^*$ be M^* and that for $\mathcal{D}$ be $M \supset M^*$. Suppose the first marker considered by the algorithm is x. There are two cases.

The first case is when $x \in M - M^*$. In this case the marker x belongs to two hypergraphs G_1 and G_2 both belonging to some $\mathcal{D}_i$ where H_i is a solid polygon. The algorithm will glue G_1 and G_2 and from Lemma 4.8, this gives a smaller standard decomposition $\mathcal{D}_i'$ of H_i.

The second case is when the marker $x \in M^*$. Let x belong to two hypergraphs G_1 and G_2 where $G_1 \in \mathcal{D}_i$ and $G_2 \in \mathcal{D}_j$ where $i \neq j$. In this case we claim that the algorithm will not glue G_1 and G_2 since gluing them would not result in a standard decomposition. To see this, let $\mathcal{D}'$ be obtained through gluing of G_1 and G_2. The split induced by x is in $\mathcal{D}^*$ but not $\mathcal{D}'$. Because the splits induced by $\mathcal{D}^*$ is not a subset of splits induced by $\mathcal{D}'$, $\mathcal{D}'$ is not a refinement of $\mathcal{D}^*$. However, as we noted earlier, every standard decomposition is a refinement of $\mathcal{D}^*$. Hence $\mathcal{D}'$ is not a standard decomposition.

From the two cases we see that no marker in M^* results in a gluing and every marker in $M - M^*$ results in a gluing. Thus the algorithm after processing $\mathcal{D}$ outputs $\mathcal{D}^*$. This yields the following theorem.

Theorem 4.10 *A canonical decomposition can be computed in $O(np)$ time given a prime decomposition.*

Next we describe an $O(np + n^2 \log n)$ time algorithm to compute a prime decomposition.

4.4 Computing a prime decomposition

We assume there exists an efficient *split oracle*. Given a hypergraph H and the value of the mincut λ, the split oracle finds a split in H or returns a pair $\{s,t\}$, such that there is no s-t split in H. In the latter case we would like to recurse on the hypergraph obtained by contracting $\{s,t\}$ into a single vertex. In order to recover the solution, we define how we can uncontract the contracted vertices.

```
PRIME(H, λ)
  if |V(H)| ≥ 4
      x ← a new marker vertex
      query the split oracle with H and λ
      if oracle returns a split (S, V(H) − S)
          {H_1, H_2} ← REFINE(H, (S, V(H) − S), x)
          return PRIME(H_1, λ) ∪ PRIME(H_2, λ)
      else the oracle returns {s, t}
          D′ ← PRIME(H/{s, t}, λ), {s, t} contracts to v_{s,t}
          G′ ← the member of D′ that contains v_{s,t}
          G ← uncontract v_{s,t} in G′ with respect to H
          if ({s, t}, V(G) − {s, t}) is a split in G
              {G_1, G_2} ← refinement of G induced by {s, t}
              D ← (D′ − {G′}) ∪ {G_1, G_2}
          else
              D ← (D′ − {G′}) ∪ G
          return D
  else
      return {H}
```

Figure 4.1: The algorithm for computing a prime decomposition.

Definition Consider a hypergraph H. Let $H' = H/\{s,t\}$, where $\{s,t\}$ is contracted to vertex $v_{\{s,t\}}$. Let G' be a ϕ'-contraction of H' such that $\phi'(v_{\{s,t\}}) = v_{\{s,t\}}$. We define uncontracting $v_{\{s,t\}}$ in G' with respect to H as a graph G obtained from a ϕ-contraction of H, where ϕ is defined as

$$\phi(v) = \begin{cases} \phi'(v) & \text{if } v \notin \{s,t\} \\ v & otherwise \end{cases}$$

See Figure 4.1 for a simple recursive algorithm that computes a prime decomposition based on the split oracle. The following lemma justifies the soundness of recursing on the contracted hypergraph when there is no s-t split.

Lemma 4.11 *Suppose H is a hypergraph with no s-t split for some $s,t \in V(H)$. Let $H' = H/\{s,t\}$, where $\{s,t\}$ is contracted to vertex $v_{\{s,t\}}$. Let $\mathcal{D}'$ be a prime decomposition of H', and let $G' \in \mathcal{D}'$ such that G' contains vertex $v_{\{s,t\}}$. And let G be obtained through uncontracting $v_{\{s,t\}}$ in G' with respect to H.*

1. *Suppose $\{s,t\}$ defines a split in G and let $\{G_1, G_2\}$ be a simple refinement of G based on this split. Then $\mathcal{D} = (\mathcal{D}' - \{G'\}) \cup \{G_1, G_2\}$ is a prime decomposition of H.*

2. *If $\{s,t\}$ does not define a split in G then $\mathcal{D} = (\mathcal{D}' - \{G'\}) \cup \{G\}$ is a prime decomposition of H.*

Proof: Every split in H' is a split in H. Therefore $(\mathcal{D}' - \{G'\}) \cup \{G\}$ is a decomposition of H. Other than G, all other elements in $(\mathcal{D}' - \{G'\}) \cup \{G\}$ are prime.

If G is not prime, then there is a split. There is no s-t split in G because H does not have any s-t split. Any split in G must have the form $(A, V(G) - A)$ where $\{s,t\} \subseteq A$. If $A \neq \{s,t\}$, then there exist some other vertex $v \in A$, which implies $|A - \{s,t\} \cup \{v_{\{s,t\}}\}| \geq 2$, and $(A - \{s,t\} \cup \{v_{\{s,t\}}\}, V(G') - A)$ is a split in G', a contradiction to the fact that G' is prime. Hence $(\{s,t\}, V(G)-\{s,t\})$ is the unique split in G. Therefore the simple refinement of G based on this unique split are both prime, and we reach the first case.

If G is prime, then we are done, as we reach the second case. □

Theorem 4.12 PRIME(H,λ) *outputs a prime decomposition in* $O(n(p+T(n,m,p)))$ *time. Where* $T(n,m,p)$ *is the time to query split oracle with a hypergraph of* n *vertices,* m *edges and sum of degree* p.

Proof: Using induction and Lemma 4.11, the correctness of the algorithm is clear. PRIME is called at most $2n$ times, and each call takes $O(p+T(n,m,p))$ time. □

Using the split oracle from Theorem 4.3 we obtain the following corollary.

Corollary 4.13 *A prime decomposition of a capacitated hypergraph can be computed in* $O(np+n^2\log n)$ *time. For uncapacitated hypergraphs it can be computed* $O(np)$ *time.*

4.5 Reducing space usage

Our description of computing the prime and canonical decompositions did not focus on the space usage. A naive implementation can use $\Omega(np)$ space if we store each hypergraph in the decomposition explicitly. Here we briefly describe how one can reduce the space usage to $O(p)$ by storing a decomposition implicitly via a *decomposition tree*.

Consider a decomposition $\mathcal{D} = \{H_1,\ldots,H_k\}$ of $H=(V,E)$. We associate a decomposition tree $T=(A,F)$ with $\mathcal{D}$ where $A=\{a_1,\ldots,a_k\}$, one node per hypergraph in $\mathcal{D}$; there is an edge $a_i a_j \in F$ iff H_i and H_j share a marker vertex. With each a_i we also store $V(H_i)$ which includes the marker vertices and some vertices from $V(H)$. This is stored in a map $\psi : A \to \cup_i V(H_i)$. It is easy to see that the total storage for the tree and storing the vertex sets is $O(n)$; a marker vertex appears in exactly two of the hypergraphs of a decomposition and a vertex of H in exactly one of the hypergraphs in the decomposition.

Given the decomposition tree T and ψ and a node $a_i \in A$, we can recover the hypergraph H_i (essentially the edges of H_i since we store the vertex sets explicitly) associated with a node a_i in $O(p)$ time. For each edge e incident to a_i in T, let C_e be the component of $T-e$ that does not contain a_i. $V(H)\cap(\cup_{a_j\in C_e}\psi(a_j))$ are the set of vertices in H which are contracted to a single marker vertex in H_i corresponding to the edge e. We collect all this contraction information and then apply the contraction to the original hypergraph H to recover the edge set of H_i. It is easy to see that this can be done in $O(p)$ time.

4.6 Hypercactus representation

For a hypergraph H, a hypercactus representation is a hypercactus H^* and a function $\phi : V(H) \to V(H^*)$ such that for all $S \subseteq V(H)$, S is a mincut in H if and only if $\phi(S)$ is a mincut in H^*. This is a generalization of the cactus representation when H is a graph.

Note the similarity of Theorem 4.5 and the definition of the hypercactus representation. It is natural to ask if there is a hypercactus representation that is essentially a canonical decomposition. Indeed, given the canonical decomposition of H, Cheng showed that one can construct a "structure hypergraph" that captures all mincuts [6], which Fleiner and Jordan later point out is a hypercactus representation [10]. The process to construct such a hypercactus representation from a canonical decomposition is simple. We describe the details for the sake of completeness.

Assume without loss of generality that $\lambda(H)=1$. We construct a hypercactus if the hypergraph is prime or a solid polygon. If H is a solid polygon, then it consists of a cycle and a hyperedge containing all the vertices. If the cycle has non-zero capacity, let H^* to be H with the hyperedge containing all the vertices removed, and assign a capacity of $\frac{1}{2}$ to each edge of the cycle. If the cycle has zero capacity, then let H^* to be a single hyperedge containing all vertices, the hyperedge has capacity 1. In both cases H^* together with the the identity function on $V(H)$ forms a hypercactus representation for H. If H is prime, let V' be the set of vertices that induce a trivial mincut, i.e. $v \in V'$ iff $\{v\}$ is a mincut in H. Introduce a new vertex v_H, and let $H^*=(\{v_H\}\cup V', \{\{v_H,v'\}\,|\,v'\in V'\})$, with capacity 1 for each edge; in other words we create a star with center v_H and leaves in V'. Define $\phi : V(H)\to V(H^*)$ as

$$\phi(u) = \begin{cases} u & u \in V' \\ v_H & u \notin V' \end{cases}$$

Then H^* and ϕ form a hypercactus representation.

For the more general case, let $\mathcal{D}^* = \{H_1, \ldots, H_k\}$ be the canonical decomposition of H. For each i, construct hypercactus representation (H_i^*, ϕ_i) of H_i as described earlier. We observe that that if x is a marker vertex in H_i, then it is also present in H_i^*. If H_i is a solid polygon this is true because $V(H_i) = V(H_i^*)$. If H_i is prime, then every marker vertex induces a trivial mincut in H_i, hence also preserved in H_i^*. Construct H^* from $H_1^*, \ldots, H_k^*$ by identifying marker vertices. This also gives us $\phi : V(H) \to V(H^*)$ by gluing together $\phi_1, \ldots, \phi_k$ naturally. (H^*, ϕ) is the desired hypercactus representation.

The construction takes $O(np)$ time and $O(p)$ space.

Theorem 4.14 *A hypercactus representation of a capacitated hypergraph can be found in $O(n(p + n\log n))$ time and $O(p)$ space for capacitated hypergraphs, and in $O(np)$ time for uncapacitated hypergraphs.*

Proof: We combine Theorem 4.10 and Corollary 4.13. The space usage can be made $O(p)$ based on the discussion in Section section 4.5. $\square$

If H is a graph, the hypercactus representation constructed is a cactus representation. Theorem 4.14 matches the best known algorithm for cactus representation construction of graphs in both time and space [32], and is conceptually simpler.

Via sparsification we obtain a faster algorithm for uncapacitated hypergraphs.

Theorem 4.15 *A hypercactus representation of an uncapacitated hypergraph can be found in $O(p + \lambda n^2)$ time and $O(p)$ space.*

Proof: Find the mincut value λ, and a $(\lambda + 1)$-sparsifier H' of H in $O(p + \lambda n^2)$ time. Theorem 3.5 shows that every mincut in H is a mincut in H', and vice versa. Therefore the hypercactus for H' is a hypercactus for H. Apply Theorem 4.14 to H'. $\square$

5 Near-linear time $(2 + \varepsilon)$ approximation for mincut

Matula gave an elegant use of MA-ordering to obtain a $(2 + \varepsilon)$-approximation for the mincut in an uncapacitated undirected graph in $O(m/\varepsilon)$ time [31]. Implicit in his paper is an algorithm that gives a $(2 + \varepsilon)$-approximation for capacitated graphs in $O(\frac{1}{\varepsilon}(m\log n + n\log^2 n))$ time; this was explicitly pointed out by Karger [20]. Here we extend Matula's idea to hypergraphs. We describe an algorithm that outputs a $(2 + \varepsilon)$-approximation for hypergraph mincut in $O(\frac{1}{\varepsilon}(p\log n + n\log^2 n))$ time for capacitated case, and in $O(p/\varepsilon)$ time for the uncapacitated case.

We will assume without loss of generality that there is no edge that contains all the nodes of the given hypergraph. Let $v_1, \ldots, v_n$ be a MA-ordering of the given capacitated hypergraph H. Given a non-negative number α, a set of consecutive vertices in the ordering $v_a, v_{a+1}, \ldots, v_b$ where $a \leq b$ is called α-tight if $d(V_i, v_{i+1}) \geq \alpha$ for all $a \leq i < b$. The *maximal* α-tight sets partition V. We obtain a new hypergraph by contracting each maximal α-tight set into a single vertex. Edges that become singletons in the contraction are discarded. We call the contracted hypergraph an α-contraction. Note that the contraction depends both on α and the specific MA-ordering.

One important aspect of α-contraction is that the resulting hypergraph has sum-degree at most $2\alpha n$ which allows for sparsifying the hypergraph by appropriate choice of α.

Lemma 5.1 *Let H' be an α-contraction of a given hypergraph H. Then* $\mathsf{sum\text{-}deg}(H') \leq 2\alpha n$ *where n is the number of nodes of H.*

Proof: Assume the α-tight partition of V is $X_1, \ldots, X_h$ where the ordering of the parts is induced by the MA-ordering. For $1 \leq i \leq h$, let $A_i = \bigcup_{j=1}^{i} X_j$. Since each X_i is a maximal α-tight set, $d(A_i, x) < \alpha$ for all $x \in X_{i+1}$. Let E' be the set of edges in H'. For a given edge $e \in H$ let e' be the corresponding edge in H'. Note that $|e'| \geq 2$ and

```
APPROXIMATE-MINCUT(H)
  if (|V(H)| ≥ 2)
      δ = min_{v∈V} δ(v)
      if (δ = 0)
          return 0
      α ← (1/(2+ε)) δ
      Compute MA-ordering of H
      H' ← α-contraction of H
      λ' ← APPROXIMATE-MINCUT(H')
      return min(δ, λ')
  else
      return ∞
```

Figure 5.1: Description of $(2+\varepsilon)$-approximation algorithm. It is easy to remove the recursion.

$c(e') = c(e)$ since the capacity is unchanged. We have the following set of inequalities:

$$\begin{aligned}
\textsf{sum-deg}(H') &= \sum_{e' \in E'} c(e')|e'| \\
&\le 2 \sum_{e' \in E'} c(e')(|e'|-1) \\
&= 2 \sum_{i=1}^{h-1} d(A_i, X_{i+1}) \\
&\le 2 \sum_{i=1}^{h-1} \sum_{x \in X_{i+1}} d(A_i, x) \\
&< 2 \sum_{i=1}^{h-1} \alpha |X_{i+1}| \le 2\alpha n.
\end{aligned}$$

□

The second important property of α-contraction is captured by the following lemma.

Lemma 5.2 *If v_i and v_j are in a α-tight set then $\lambda(v_i, v_j) \ge \alpha$.*

Proof: Assume without loss of generality that $i < j$. Consider any k such that $i \le k < j$. We have $d(V_k, v_{k+1}) \ge \alpha$ because i and j are in the same α-tight set. By Lemma 2.2, $\lambda(v_k, v_{k+1}) \ge d(V_k, v_{k+1}) \ge \alpha$. By induction and using the fact that for any $a, b, c \in V$, $\lambda(a,c) \ge \min(\lambda(a,b), \lambda(b,c))$, we have $\lambda(v_i, v_j) \ge \alpha$. □

Figure 5.1 describes a simple recursive algorithm for finding an approximate mincut.

Theorem 5.3 APPROXIMATE-MINCUT *outputs a $(2+\varepsilon)$-approximation to an input hypergraph H and can be implemented in $O(\varepsilon^{-1}(p + n\log n)\log \frac{n\delta(H)}{\lambda(H)}))$ time for capacitated hypergraphs and in $O(\varepsilon^{-1}p)$ time for uncapacitated hypergraphs.*

Proof: We first argue about the termination and run-time. From Lemma 5.1, $\textsf{sum-deg}(H') \le \frac{2}{2+\varepsilon} n\delta(H)$. Since $\textsf{sum-deg}(H) \ge n\delta(H)$, we see that each recursive call reduces the sum degree by a factor of $\frac{2}{2+\varepsilon}$. This ensures termination.

If the hypergraph is uncapacitated, the running time of each iteration is bounded by the sum degree. The sum degree reduces by a factor of $\frac{2}{2+\varepsilon}$ in each iteration. Hence $O(\sum_i^\infty (2/(2+\varepsilon))^i p) = O(\varepsilon^{-1}p)$ upper bounds the running time for uncapacitated hypergraphs.

We now consider the case when the hypergraph is capacitated. Let H'' be any non-trivial hypergraph (that has at least two vertices) that arises in the recursion. The mincut value does not reduce by contraction and hence $\lambda(H'') \ge \lambda(H)$ which in particular implies that $\delta(H'') \ge \lambda(H)$, and hence $\textsf{sum-deg}(H'') \ge 2|V(H'')|\lambda(H)$. After the first recursive call the sum degree is at most $\frac{2}{2+\varepsilon} n\delta(H)$. Thus the total number of recursive calls is

$O(\varepsilon^{-1}\log(\frac{n\delta(H)}{\lambda(H)}))$. The work in each call is dominated by the time to compute an MA-ordering which can be done in $O(p + n\log n)$. This time gives the desired upper bound on the run-time of the algorithm for capacitated hypergraphs.

We now argue about the correctness of the algorithm which is by induction on n. It is easy to see that the algorithm correctly outputs the mincut value if $n = 1$ or if $\delta = 0$. Assume $n \geq 2$ and $\delta(H) > 0$. The number of vertices in H' is strictly less than n if $\delta(H) > 0$ since the sum degree strictly decreases. Since contraction does not reduce the minimum cut value, $\lambda(H') \geq \lambda(H)$. By induction, $\lambda(H') \leq \lambda' \leq (2+\varepsilon)\lambda(H')$. If $\lambda(H') = \lambda(H)$ then the algorithm outputs a $(2+\varepsilon)$-approximation since $\delta \geq \lambda(H)$. The more interesting case is if $\lambda(H') > \lambda(H)$. This implies that there are two distinct nodes x and y in H such that $\lambda(x, y, H) = \lambda(H)$ and x and y are contracted together in the α-contraction. By Lemma 5.2, $\lambda(x, y, H) \geq \alpha = \frac{1}{2+\varepsilon}\delta$ which implies that $\delta \leq (2+\varepsilon)\lambda(H)$. Since the algorithm returns $\min(\delta, \lambda')$ we have that the output is no more than $(2+\varepsilon)\lambda(H)$.

□

Since δ can be much larger than λ in a capacitated hypergraph, we can preprocess the hypergraph to reduce δ to at most $n\lambda$ to obtain a strongly polynomial run time.

Lemma 5.4 *Let $\beta = \min_{i>1} d(V_{i-1}, v_i)$ for a given MA-ordering $v_1, \ldots, v_n$ of a capacitated hypergraph H. Then $\beta \leq \lambda(H) \leq n\beta$.*

Proof: From Lemma 5.2, $\lambda(u, v) \geq \beta$ for all $u, v \in V$ because V is a β-tight set. Therefore $\lambda(H) \geq \beta$. Let $i^* = \arg\min_{i>1} d(V_{i-1}, v_i)$. Then,

$$d(V_{i^*-1}, V \setminus V_{i^*-1}) \leq \sum_{j=i^*}^{n} d(V_{i^*-1}, v_j) \leq (n+1-i^*)\beta \leq n\beta.$$

Thus, the cut $(V_{i^*-1}, V \setminus V_{i^*-1})$ has capacity at most $n\beta$, and hence $\lambda(H) \leq n\beta$. □

Let β be the value in Lemma 5.4, then a $2n\beta$-contraction of H yields a non-trivial hypergraph H' where $\mathsf{sum\text{-}deg}(H') = O(n^2\beta)$. This also implies that $\delta(H') = O(n^2\beta)$. Applying the $(2+\varepsilon)$ approximation algorithm to H' gives us the following corollary.

Corollary 5.5 *A $(2+\varepsilon)$ approximation for hypergraph mincut can be computed in $O(\varepsilon^{-1}(p + n\log n)\log n)$ time for capacitated hypergraphs, and in $O(\varepsilon^{-1}p)$ time for uncapacitated hypergraphs.*

Remark Suppose we use the Queyranne ordering instead of MA-ordering, and define $v_a, \ldots, v_b$ to be α-tight if $\frac{1}{2}(d(V_i, v_{i+1}) + d'(V_i, v_{i+1})) \geq \alpha$ for $a \leq i < b$. The algorithm in Figure 5.1 produces a $(2+\varepsilon)$-approximation with this modification.

6 Concluding Remarks

We close with some open problems. The main one is to find an algorithm for hypergraph mincut that is faster than the current one that runs in $O(np + n^2\log n)$ time. We do not know a better deterministic run-time even when specialized to graphs. However we have a randomized near-linear time algorithm for graphs [21]. Can Karger's algorithm be extended to hypergraphs with fixed rank r? Recently there have been several fast s-t max-flow algorithms for undirected and directed graphs. The algorithms for directed graphs [28, 29] have straight forward implications for hypergraphs s-t cut computation via the equivalent digraph. However, hypergraphs have additional structure and it may be feasible to find faster (approximate) algorithms.

We described a linear-time algorithm to find a maximum-flow between the last two vertices of a tight-ordering of a hypergraph (the flow is in the equivalent digraph of the hypergraph). We believe that such a linear-time algorithm is also feasible for the last two vertices of an MA-ordering of a hypergraph. Some of the research in this paper was inspired by work on element connectivity and we refer the reader to [4] for related open problems.

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